THE GEOMETRY

OF THE

THREE FIRST BOOKS OF EUCLID,

BY DIRECT PROOF

FROM DEFINITIONS ALONE.

BY

HENSLEIGH WEDGWOOD, M.A.

FOR THE PEOPLE.

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iiled to supply the want felt by a large number teachers in public and private schools of a Class Book for Junior Students. purpose is to convey, in clear and concise terms, general notions of all principal divisions of Physical Science, illustrated copiously by diagrams, owing the forms and arrangement of apparatus, and the manner of performing the most important experiments.

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WITH

AN INTRODUCTION ON THE PRINCIPLES
OF THE SCIENCE.

BY

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PREFACE.

The attempts at a reformation of the Premises of Geometry have been so numerous, and have met with so little success, that another essay in the same direction will doubtless be classed by many with the endeavours to square the circle or find perpetual motion. A little consideration, however, will shew that the circumstances are widely different. The notion of irrational quantities, or quantities whose proportions cannot be exactly expressed by means of numbers, is one which causes difficulty only to the uninstructed. However extended the numbers of a fraction may be by which we attempt to express a proportion, it is readily seen, after a little familiarity with arithmetical conceptions, that the numerator may be a little too great or too small, while the addition or subtraction of an unit may make too great a difference in the opposite direction. There is then no reason to expect that any particular proportion, as that between the circumference and the diameter of a circle, should be capable of exact numerical expression; or, in other words, that the squaring of the circle should be a possible problem. In geometry, on the other hand, there is positive *à priori*

argument for the possibility of attaining the end which reformers have had in view, and nothing but his predecessors' want of success to discourage the efforts of any new competitor in the same arena. The figures which form the subject of geometrical reasoning being wholly the creation of the understanding, it would seem that they can be endowed with no essential qualities except such as are derived from the plan on which the elements of figure are put together in the conception of the geometrical species. If, therefore, we were able to analyse the first step taken by the understanding in the conception of figure, and to indicate the immediate relation by which the ultimate elements of surface and of line are combined in the conception of the simplest species of figure, the propositions enouncing this primitive synthesis, together with those laying down, in like manner, the composition of the more complicated species, should constitute premises, from whence might directly be deduced every possible relation of the geometrical system. The question then arises, Have any of the proposed amendments been based on the ultimate analysis of all the species of geometrical figure?—and specially, Has the true analysis of a plane as yet been propounded?

In the complete conception of every kind of surface, each infinitesimal element of the surface must be brought successively before the mind and arranged in proper relation to the rest of the system, and whatever can be distinctly conceived

may be expressed in language. It must, therefore be inherently possible to express in words the principle of arrangement or relation between its ultimate parts, characteristic of a plane as well as of every other species of surface. It was by such considerations that the author was led to disregard the old argument, that if the thing could be done at all, it would have been done long ago; but, as soon as he began to study the analysis of figure, he found that the previous question, by what intellectual process we are originally made acquainted with figure in general, which was necessary in order to determine what was, and what was not an elementary conception, was entirely unsettled. It thus became necessary to undertake the examination of one of the most vexed questions of metaphysics, and to trace the course of action and complex exercise of our faculties, by which we originally obtain the knowledge of body, space and form.* Having carefully gone through this inquiry, and obtained certain results to his own satisfaction, the author felt it a strong corroboration of the solidity of his groundwork, when he found that the definitions to which he was led by the metaphysical investigation, including one wholly unexpected of a plane, afforded an adequate basis for the science of

* "Principles of Geometrical Demonstration," Taylor & Walton, 1844. "On the Development of the Understanding," 1848. "On the Knowledge of Body and Space." "Trans. Cambridge Phil. Soc.." Vol. ix., 1850.

geometry, enabling us to dispense as well with the axioms, as with all *ex absurdo* proof, which has always been regarded as an incongruity in the system.

As the only effective test of the actual attainment of the end which has so long been had in view, the system proposed is applied in the following pages to the geometry of the first three books of Euclid, marking those propositions which are simply copied out without any material alteration in the proof.

If there be no important fallacy in the reasoning of the following pages, the premises adopted in our system are not merely an improvement on those in ordinary use, but they are the ultimate expression of the mode in which the fundamental conceptions of the science are brought into intellectual existence, and must therefore be the primary source from whence all geometrical conviction is derived. No further room will then be left for essential reform, and it would be contrary to the spirit of sound philosophy if the name of Euclid were weighty enough to preserve the sway of his imperfect system in English education, when once the true foundation of the science was effectually made known.

ERRATA.

Page.	Line.	
46	14	*for* D C B *read* E C B.
50	1	*after* D F *insert*, the corresponding side of the other triangle.
56	1	*for* 24 *read* 23.
69	14	*for* 31 *read* 30.
74	10	*after* X L *insert* Euclid I. 38.
75	13, 14	*for* E F *read* D F.
103	18	*for* D E *read* D C.

INTRODUCTION.

Geometry is the science of form, position, and magnitude, the subject of which it treats consisting of figures drawn according to some definite law, while the aim of the science is the determination of relations of position and magnitude necessarily holding good between different parts of the figured system, though not expressly mentioned in the rule by which the latter is originally defined in the apprehension of the student. Thus, for example, the simplest kinds of figure are the straight line and the plane, and accordingly rectilineal figures, or figures constructed of straight lines and plane surfaces (and primarily the triangle as the rectilineal figure of fewest sides), form the earliest subject of geometrical investigation. Now the form of a triangle may be varied at pleasure, by changing the proportion between the sides, without necessarily raising the question, whether there be any corresponding variation in the proportion of the angles. We may imagine a triangle

of three equal sides, or a triangle in which one of the sides is much greater than either of the others, without, in the first place, considering whether the angles will be equal or unequal, or which of them will be the greater; but geometry teaches us, that if two sides of a triangle are equal, the angles opposite to the equal sides are also equal to each other; and if the sides are unequal, the angle which is opposite to the greater side will be greater than the angle which is opposite to the less.

The principles of most obvious authority in reasoning are the propositions laying down the sense in which the terms of the demonstration are to be understood. The student who uses a certain term to signify the conception expressed in detail in a given expression, will perceive, that every actual example of the thing signified must necessarily be possessed of the characteristics mentioned in the defining expression, because it is only by the possession of those characteristics that an actual object can earn a title to the designation in question. If I use the word *triangle* to signify a rectilineal figure of three sides, I can only recognise a particular figure as a triangle by the apprehension of the three straight lines of which it is composed; and, accordingly, I perceive that every triangle must necessarily be bounded by three straight lines.

Thus it is, that every definition rightly under-

stood assumes the form of a necessary truth, or of a mere truism, in case the thing signified by the term defined (as in the foregoing example) is of such a nature that it cannot be made the object of contemplation without the distinct recognition of the analysis enounced in the definition; and if the premises in our systems of geometry had been composed exclusively of propositions owing their authority to such a principle, the necessity of the conclusions would have been involved in none of that mystery which has been so fertile a source of speculation. Hitherto, however, geometers have not succeeded in laying an adequate foundation of the science in definitions alone. It has always been found necessary, either openly or covertly, to call in the aid of axioms, or propositions, the truth of which we find ourselves compelled, after more or less reflection, to admit, although we may be unable to explain the intellectual process by which our assent is extorted.

In justification of the appeal to an authority of such a nature, the axioms are commonly spoken of as self-evident truths, to which appellation their claim has not been very clearly expounded. A self-evident proposition ought to carry conviction on the face of it irresistible to all who rightly understand the terms of the proposition, and this can only be the case when the correct conception of the subject (as in definitions) involves the re-

cognition of the features constituting the predicate of the proposition. To perceive the necessary truth of the proposition, that "if two straight lines meeting a third, make the two internal angles less than two right angles, the two straight lines shall meet if produced far enough" (the axiom of Euclid relating to parallel lines), requires an effort of the understanding essentially differing from the mere comprehension of the meaning of the proposition; and the axiom is probably at the outset accepted by a large proportion of students on the authority of the teacher without any clear apprehension of the evidence of the assertion. Before the geometer is contented to rest his system upon principles of whose authority he is able to render so little account, he ought to be thoroughly satisfied that he has exhausted the resources of definition, that his premises exhibit the ultimate analysis of the conceptions concerning which he proposes to reason, or their original construction out of the elementary materials of thought.

It requires little consideration to show, that such a limit is far from being attained in the ordinary system of geometry. It is a sufficient proof of shortcoming, that it contains no effective definition of a straight line. The assertion, that a straight line is "a line lying evenly between its extreme points," amounts to no more than this, that it is a line lying straight between its extreme

points; and as a proposition so manifestly identical can lead to no real advance in reasoning, the definition is never afterwards referred to, and forms no part of the real premises of the system.

The definitions of parallel straight lines, and of a plane surface, are as follows:

Parallel straight lines are such as are in the same plane, and being produced ever so far both ways do not meet.

A plane surface is that in which any two points being taken, the straight line between them lies wholly within such surface.

In neither of these cases does the definition exhibit a simple analysis of the essential meaning of the term defined. We can distinctly imagine a pair of parallel straight lines, or a plane surface, without a thought in our minds of the indefinite prolongation of the lines in the one case, or of the system of straight lines joining every separate pair of points in the plane, in the other case. We apprehend the planeness of a surface by passing our hand over it in a track, of which it is possible, that no portion may consist of a single straight line. The geometrical figure is in neither case defined by the relations of its own essential elements, but by conditions involving a reference to some external system, the notion of which necessarily presupposes the distinct conception of the figure under definition. We must plainly be

acquainted with parallel lines as a system of certain figure, before we can recognise the fact, that lines in such a position may be prolonged for ever without meeting. In like manner, the form of a plane surface must previously be known as a substantive object of thought, in order to supply us with the system of straight lines by coincidence with which the *planeness* of the same surface is to be established under this definition.

The analysis, then, of what is fundamentally meant by the attributes of planeness, or of parallelism, as well as of straightness, is to be sought for in other quarters. In this research it must be borne in mind, that figure is considered in geometry as extended in empty space, and therefore, as marked exclusively by position, the only character by which the parts of space are distinguished. A certain point will be a point occupying a definite position in space. A figure will be conceived as a line or a surface, extending through a succession of points arranged in a certain scheme of relative position, the enunciation of which will be the object to be aimed at in definition, and the analysis will be pushed to its utmost limits, when the definition expresses the fundamental relation of each individual point in the figure to its immediate neighbours, and consequently, to the remainder of the system.

What the fundamental relations of position

are, I have elsewhere endeavoured to shew, from a careful examination of the active process by which the knowledge of space and all the relations which it involves are originally acquired. But without at present entering upon the metaphysical inquiry, it will be seen, that the position of an object is given in the knowledge of a track by which it may be reached from a known station. When the organ employed in reaching the object is conceived as an individual point, the station from whence the motion commences, as well as the position attained at any moment, will both be single points, while the track of motion will be a mathematical line. Thus the position of a point is determined by the nature of the line by which it is united to a point antecedently known, and the identity of points is accordingly proved in geometry, by shewing the coincidence or entire identity of the lines by which the points in question are united with the same given point.

Now motion, in as far as the relations of space are concerned, admits of variation in two ways, each giving rise to the conception of an elementary attribute, or one, which can only be explained by reference to the various phases exhibited in actual existence, in the same way that colour can only be explained, as the attribute, of which white, blue, red, etc., are particular phases.

The elementary attributes of motion (and there-

fore of a line, as the track of a point in motion) are: first, the longitudinal extent of the track, which keeps continually increasing from the commencement of the motion; and secondly, the direction of the motion at any given instant, a character admitting of variation in each infinitesimal element into which the track may be divided. The nature of the entire track or figure of the line will be conceived by combining in a single act of thought the continuous succession of infinitesimal elements, each with their distinctive character of direction and distance from the origin. Thus, the position of a point will be determined by the character, in respect of distance and direction, of a line by which it is united with a point already known. But the same fixed point may be attained from a given station by tracks of a wholly different description; the same two points may be united by lines which have nothing in common except the beginning and the end. The very conception of a triangle A B C, implies the possibility of recognising the identity of the point C attained by motion from a given station A either through the track A B, B C, or straight through A C. That is to say, the aggregate character of the broken line A B, B C, in respect of distance and direction, must be recognised as equivalent, in the determination of position, to that of the straight line A C. There must then be some fundamental

connection between distance and direction, some means of reducing distance in different directions to a common standard, in order to render possible the equivalence, in the determination of position, of different combinations of those elements.

The discrimination of the infinite variety of directions in which motion is possible from any point in space, is based on the two fundamental relations of opposition and transverseness. If, after moving through a certain distance in any one direction, we stop to contemplate, from the station so attained, the position of the point from whence we started, it will be conceived as lying at the distance traversed in attaining our present station, and in a direction, the relation of which to that of the original motion is designated by the term opposition. In other words, an object which has moved through a certain distance in a given direction will be brought back, by the same extent of motion in the opposite direction, to the position originally occupied: and the fundamental characteristic of the relation will be, that motion in a given direction is exactly nullified, in the determination of position, by the same extent of motion in the opposite direction. Thus a direction and the one opposed to it may be considered as the positive and negative modifications of a common direction.

The origin of the notion of transverseness was

traced, in the enquiry above alluded to, to the motion of the hand along a smooth surface sensibly approaching a plane; and it was shown, that the agent, in the course of such an experiment, would be conscious of being able to move freely in a multiplicity of directions along the surface of the body at any point, while, at the same time, he would be cognizant of an absolute resistance to motion in a certain direction subsequently known as the normal to the surface at the point in question. Thus he would have experience of a certain direction, so related to a multiplicity of others, that motion in any of the latter is compatible with a total absence of motion in the former direction, either positively or negatively considered. Of directions thus related, each of those in which freedom of motion is left along the surface, is said to be transverse to the direction in which all motion is simultaneously forbidden by the resistance of the body.

Now, if motion in a direction C A be compatible with a total absence of motion in a direction C B, it can only be because distance in the latter direction is essentially independent of distance in the former, and, therefore, motion in the direction C B must equally be compatible with a total absence of motion in the direction C A. In other words, if one direction is transverse to a second, the second is transverse to the

first, or the relation is reciprocal between the two.

We learn, in the next place, that motion in two transverse directions essentially constitutes motion in a third direction, conceived as intermediate between the former two. The cognizance of opposition to action, implies a consciousness of ability to perform the action resisted if the external obstacle were removed. Thus the resistance of which we are cognizant while moving our hand over the surface of a body, at the same time that it makes known to us the actual absence of motion in the direction of the normal, will suggest the possibility of motion in that direction, without further change in the conditions of the experiment than the removal of the bodily obstacle; that is to say, the same experience which made known to us the relation of transverseness, will lead us to conceive the possibility, in empty space, of advancing in the direction of the normal, without ceasing to move in the lateral direction in which the surface formerly extended, or in other words, of moving simultaneously in a given direction and in one transverse to it.

But when an object moving simultaneously in two directions is contemplated from without the sphere of the influences to which the separate movements are owing, it will appear to move in a third direction, related more or less nearly to the

direction of either of the component movements, according as the distance traversed in that direction is greater or less than the distance simultaneously traversed in the other.

An agent carried along in a certain direction by the motion of the rigid system in which he is placed, as in the cabin of a ship for instance, has the same freedom of action within the limits of the system as if the latter were at rest. He is capable of moving with the same facility in a direction transverse to that of the ship's motion as in the same direction with it. But if the real direction of his motion could be observed, with respect to external objects, while he is moving across the cabin, it would be found to lie in some intermediate direction, standing in a closer relation to that of the ship's motion or the transverse direction, according as the rate of the ship's motion is more or less rapid than that of his own walk across the cabin.

Thus, if an object be supposed to move independently in two directions, C A, C B, transverse to each other, and C P be the real direction of the motion in space, the relation of C P to C A and C B may be expressed by reference to the proportion in which motion in the direction C P admits of resolution in the directions C A and C B respectively. Now, let either of these directions as C A (Fig. 1) be taken as a standard, and let motion in

the direction C A be compounded with an indefinitely small proportion of motion in the direction C B. The result will be motion in a direction C P, differing extremely little from C A. Then by continually increasing the proportion of motion in the direction C B, and diminishing that in the direction C A, we shall obtain a series of intermediate directions varying in their relation to C A in every degree from coincidence to transverseness. In like manner, the combination of motion in the direction C B with C D, the opposite of the original standard, will furnish a similar series of directions intermediate between C B and C D, which will appear as the continuation of the former series. Beyond C D, on the other side, the series may be carried on by combination with motion in the direction of C E, the opposite of C B, and it will finally be brought back to the point from whence we started by combination of motion in the direction C E with motion in the original direction C A.

Thus, by combination of motion in a single direction, positively and negatively considered, and in a transverse direction taken in like manner in a positive and negative sense, we are enabled to distinguish a continuous circle of directions, each of which is placed between two neighbours differing from it by an indefinitely small amount of variation in opposite directions.

Now, the relation of transverseness being (as we have seen) primarily known as a relation of one direction to a multiplicity of others, let C A, C P (Fig. 1) be any two directions to both of which a third, C F, is transverse. Then motion in the direction C P may be resolved in a direction C A, and a direction transverse to it, C B. But as the directions C P and C A are both transverse to C F, motion in either of those directions will be wholly devoid of motion in the direction C F; that is to say, that motion in the direction C P, as well as one of the elements into which it may be resolved, will be wholly without effect in the direction C F, and, therefore, the remaining element, or the motion in the direction C B, must be equally ineffective in the same direction, or C B also will be transverse to C F. But motion in the directions C A and C B being both ineffective in the direction C F, the same must be true of every motion compounded of those elements; or in other words, every direction intermediate between C A and C B and their opposites will be included in the series transverse to C F. Thus it appears that if, in the series of directions C A, C P, etc., transverse to a given normal or standard direction C F, any individual, as C A, be taken as a second standard, the series will include a direction C B transverse to C A (and therefore constituting a third standard transverse to each of the other two),

together with every direction intermediate between C A and C B, positively and negatively considered.

Now let each individual of the series CA, CP, etc., arising from the combination of CA and CB, again be combined with the direction CF transverse to them all; the result will be a succession of series of the same kind, the aggregate of which will embrace every direction diverging from the point C, throughout an entire hemisphere of space. In like manner, the combination of the same primary series with the opposite to CF, will give the directions of the opposite hemisphere. Thus the entire scheme of directions diverging from a point in space, will be constructed by the combination of a single standard direction with the succession of series arising from the combination of two other directions, transverse to the original standard and also to each other; and any particular direction may be identified by the proportion in which distance in that direction is composed of distance in each of the three transverse standards or axes. If the vertical or up and down direction, for example, be taken as the primary axis of direction, the series transverse to it will be the directions of the horizontal plane; and if we take any one of these, as the right and left direction, for our second standard, and the transverse or the fore and aft direction for our third, the position of any direction

in space may be defined by the proportion in which motion in that direction is composed of motion in the directions up and down, right and left, and fore and aft, respectively.

We have it now in our power to explain the fact formerly observed, that the same position may be determined by tracks of a wholly different description, from the same starting point. If we suppose the motion in each elementary portion of one of two different tracks from one point to another, to be resolved in the direction of the three transverse axes, the motion in the entire track will be equivalent to the aggregate motion in the direction of each of the three axes. In like manner, the motion in the other track may be resolved into a certain amount of motion in the direction of the same three axes; and, in this condition, will admit of direct comparison with the aggregate motion in the former track. When the motion in the direction of each of the three axes is of like extent in either track, the entire spaces traversed will be the same in respect of distance and direction, and the same position will be attained in both cases.

The position of a point may now be defined as the relation depending on the character of the space by which it is separated from a given point in respect of distance and direction; whence it follows, that points identical in position lie at a like distance from a point antecedently known, in

whatever direction their respective distances may be compared.

Having thus supplied the first great want in the premises of the ordinary system of geometry, by a thorough investigation of the relation of position, we shall proceed to construct definitions of the elementary species of geometrical figure on the principles established in the foregoing inquiry.

We have seen that the shape of a line or nature of the track pursued by a point in motion, depends upon the direction of the motion, or of the line at the points successively brought under notice in the apprehension or imagination of the entire line. The very conception then of linear figure, supposes the capacity of comparing the direction from one instant to another, in the track of motion. The moment we lose count of our direction, as in wandering in a wood or in the streets of a crowded city, we lose all knowledge of the track we are pursuing, as completely as if we were carried along in the cabin of a ship or in a railway carriage.

We may then suppose a point to move for any extent in the same direction; or, after moving for a certain extent in a given direction, it may be supposed to diverge for a while in a track of any other description and again to return to the original direction. In the former case, the point will move in a straight line; in the latter (neglecting

that part of the path traversed in the intermediate period), it will move first in one straight line and afterwards in a second one parallel to the first.

A straight line may, accordingly, be defined as a line lying throughout in the same direction, or a line passing through each successive point in space situate in a certain given direction from a given point. In like manner, the definition of parallel straight lines will be, straight lines lying in the same direction in a system and not forming parts of the same straight line.

Here it will be observed, that the characters of straightness and parallelism, each of them attributes of the entire line, are reduced to the single relation of identity of direction, a character of each infinitesimal element of the line, and a real advance in analysis is embodied in the proposed definitions.

If the fundamental analysis of a plane had been equally obvious, it is probable that little difficulty would have arisen respecting the validity of the former two, but as long as the necessity of resorting to premises of a description other than definitions remained, it would be open to doubt with which of the actual premises the blame of failure ought to lie. Thus the question began to occur, What is direction? Is it not simply the position of a certain straight line, and is not relative direction fundamentally measured by the angle inter-

cepted between straight lines in the directions compared? Does not, therefore, the idea of direction rest upon that of a straight line rather than *vice versâ?* Nor was there any escape from the dilemma until the notion of relative direction was placed, as in the foregoing inquiry, on a basis independent of angular magnitude.

In our system, the objection meets with a ready answer. Direction is a relation incapable of logical analysis, designating the mode in which motion admits of variation, and of which it exhibits a definite phase, at every point in the track pursued; and the relation between two directions is fundamentally measured by the proportion in which distance in the second admits of resolution into distance in the first, and in a transverse or wholly different direction respectively.

The notion of a plane is doubtless originally derived from the experience of a solid surface of uniform inclination, that is to say, a surface whose absolute resistance to motion is everywhere in the same direction. In every direction transverse to this fundamental direction, the surface may be freely traversed, while all motion in the direction of the resistance is opposed by the solid substance of the body. Thus the motion of a point along a solid plane will be limited by the sole condition of a total absence of motion in a certain given direction, and conversely, if a point be supposed

to move in a track the direction of which is everywhere transverse to a certain constant direction, it will pass through a series of positions related to each other as those successively traversed by a point moving on a solid plane. Thus a plane may be defined as a surface passing through all the the points which can be reached from a given point by motion transverse to a given direction, called the normal to the plane. From such a definition it is obvious, that the series of straight lines diverging from a point in directions transverse to a given direction will be included throughout their entire length in the same plane; and as such a series includes directions in every possible relation which one direction can bear to another, it follows, that any two straight lines meeting in a point may be included in a single plane.

Let C A be a straight line pointing to the left, C B transverse to C A, and let a moveable straight line, C P, be supposed to revolve round C in the plane of C A and C B from left to right, passing successively through every direction intermediate between C A and C B. Then the portion of plane surface intercepted between C A and C P will continually increase in magnitude in the part abutting upon the point C as the arm C P sweeps over a fresh segment of the plane in its progress from left to right. In other words, the angular distance, as it is callcd, between C A and C P, or the

the magnitude of the angle made by those lines (which is measured by the quantity of plane surface intercepted between them abutting on the point of intersection) will continually increase as the direction of the moveable arm approaches to the relation of transverseness, or as distance in the direction C P includes a larger proportion of distance in the direction C B, and a less in the direction C A. Thus the magnitude of the angle between two straight lines may be used to define their relative direction; and as the magnitude of any angular segment may always be expressed by a direct numerical ratio to that of the rectangular segment between straight lines in transverse directions (which itself constitutes one quarter of the whole superficial expanse round any point in a plane), it affords a more simple and often more convenient measure of the difference in direction than the fundamental distinction, by reference to the decomposition of distance in the direction of one arm of the angle in certain proportions into distance in the direction of the other arm, and in a direction transverse to it respectively.

The principles which govern the relations of magnitude, Equal, Greater, and Less, are laid down in the nine first axioms of Euclid, the whole of which may conveniently be replaced by a single proposition enouncing the conditions by which the equality of given dimensions, or the

superior magnitude of one or the other of them, is ultimately to be determined.

The meaning of equality, and of greater and less, is identity or excess, on one side or the other, in respect of the quantity of space occupied by the dimensions compared; and thus the relative magnitude of particular dimensions must ultimately be tried by bringing them both actually to occupy the same space, or making one of them to occupy a certain space which is wholly filled by a part of the other. And such, in fact, is the end to which all processes of measurement are directed, either by bringing the dimensions compared into actual coincidence with each other (taking them to pieces when necessary) or with a third magnitude which can readily be compared with both the original dimensions. By such means, it is made evident to sense whether they occupy the same space, or on which side is the excess. In like manner, in order to demonstrate or make evident to the understanding the relative magnitude of dimensions which are the object of that faculty, as the dimensions occupying certain positions in definite species of figure, it will suffice to shew, from the construction of the figures in which they lie, that the magnitudes compared, or their component parts, may be applied to each other, so as wholly to coincide or occupy the same space, or to coincide to the whole extent of one of the magnitudes, leaving

a portion of the other unoccupied. The result in the former case, will be to shew that dimensions of the kind in question are necessarily equal to each other; in the latter, that the dimension which wholly coincides with a part of the other, is the less of the two. Hence the proposition which is to replace the axioms of Euclid relating to equality and inequality.

Two equal magnitudes are such, that one of the two (or the sum of all its parts) may be made to coincide with the other or the sum of all its parts. But if one of the two magnitudes, or the sum of all its parts, coincide with a portion of the other, leaving a portion of the second magnitude unoccupied, the including magnitude is said to be the greater, the included the less.

The process by which it is shewn that one figure may be applied to, or made to coincide with, another, for the purpose of ascertaining their relative magnitude, is called the proof by superposition, and is, in truth, the process upon which all demonstration of equality or excess must fundamentally rest.

The nature of the reasoning is, however, not unfrequently misapprehended by beginners, and the basis of the demonstration looked on with suspicion, as if the certainty of the conclusion would thus be left to depend upon the exactitude of particular measurements. But it must be observed,

that the process does not consist in actual measurement of the figures by which the reasoning is illustrated, but in showing from the conditions of the case, that figures constructed according to a certain plan, when properly applied to each other, must necessarily either wholly coincide, or partially coincide, leaving a clear excess on the one side or the other.

Whatever may have been the reason which gave Euclid a prejudice against this necessary part of the proof, it is certain that he keeps it as much as possible in the background, and frequently resorts to circuitous reasoning and *ex absurdo* proofs, in order to demonstrate propositions, which might be proved at once by direct superposition. But nothing is added to the cogency of demonstration by the length of the deductive process; and it would surely be more philosophical in all cases to lead the student, by the shortest path, to the basis on which his conviction must ultimately rest.

The definition of proportion, in the fifth book of Euclid, is often the source of difficulty to the student, who is conscious that the very complex relation described in that proposition, is not what he understands under the name of proportion, and is embarrassed by the real inconsequence of applying the results of Euclid's demonstrations to proportional quantities in his own sense of the term. The truth appears to be, that Proportion is

an elementary relation, the discernment of which between distances observed on different occasions, originally gives rise to the conception of magnitude, as a quality of which the distances compared exhibit different phases; in the same way that the discrimination of blue and white and red gives rise to the idea of colour, as the quality embracing the whole of these phenomena. If colour consisted exclusively of white and black, one tint would differ from another only in the degree of illumination, and the case would be precisely analogous with that of magnitude, which differs only in degrees of more or less. A greater magnitude is then, fundamentally, a dimension which bears a greater proportion than a second to some common standard; or conversely, a dimension to which the common standard bears a less proportion than to a second dimension; while the second dimension is, in either case, determined by the same fundamental conditions as the less of the two magnitudes.

Thus an increase of the antecedent of two dimensions, or a diminution of the consequent, corresponds to an increase of the proportion between them, and conversely a diminution of the antecedent, or an increase of the consequent to a diminution of the proportion. It is evident then, that by making both members of a proportion greater or smaller in the same proportion, that is, by

adding to or subtracting from both members magnitude in the same proportion with that between the original members themselves, no alteration will be made in the proportion, which will be as much increased by the addition of magnitude to the antecedent, as it is diminished by the addition of proportional quantity to the consequent, and *vice versâ*. When the proportion between two dimensions or their relation in respect of magnitude is the same with that between two other dimensions, the four are elliptically said to be proportional, or to be in proportion, meaning that they are in the same or in equal proportion.

From the foregoing view of the connection between magnitude and proportion, all the propositions of the fifth book of Euclid are immediate consequences. Thus, the multiplication of both members of a proportion, or a ratio, as it is called in Euclid, is the continued addition to each of magnitude in the same proportion with the original, and the division of both members by the same quantity being the converse of the foregoing operation, must give rise to parts in the same proportion with the wholes from whence they were derived.

Again, if A be to B as C to D, and C be substituted for B in the first ratio, the like alteration must be made in the second, in order to preserve the identity of the two proportions; that is to say,

the consequent in the second ratio must also be altered in the proportion of B to C, or the antecedent in the opposite proportion of C to B. Therefore, if C be substituted for B in the first ratio, the identity of the two relations will be preserved by substituting B for C in the opposite member of the second ratio, or if A be to B as C to D, A will be to C as B to D.

The notion of magnitude is transferred from linear to superficial extension by reference to the comparative extent of motion for which scope is given by surfaces of different size, without going twice over the same ground. The magnitude of a surface is apprehended by passing the finger over the entire surface; in which operation the size of the bodily organ must evidently be taken into account, inasmuch as the motion of which a mathematical point is capable without going twice over the same ground, is infinite in extent as well in a small surface as in a large one. In reasoning, therefore, concerning the proportion of superficial magnitudes, the surfaces compared must be considered as ultimately composed of lines of finite thickness, however small. We may thus conceive rectangles as composed of a series of straight lines, each equal and parallel to the base, and all of the same thickness. Rectangles, therefore, of the same height, will be composed of the same number of parallel lines, each equal to the base of the

rectangle to which it belongs; and the magnitude of the rectangles, which is measured by the total length of line into which each rectangle can be divided, will be proportional to the length of their bases.

We have indicated in the preceding pages the chief deficiencies to be supplied and alterations to be made in the premises of Euclid. One or two considerable variations will also be found in the body of the science. In the first place, the problems are omitted as being wholly unessential to the demonstration of the theorems with which they are connected. The student has credit in the postulates of Euclid for the possession of a ruler and a pair of compasses, and whenever the proof of a proposition requires any addition to the figures mentioned in the statement of the proposition, a problem is thought necessary in order to show him the means of describing the additional figure with those implements. The only object which can be gained by such a proceeding, is either to teach the student to describe the figure with exactitude, or to show the inherent possibility of the construction. With respect to the former purpose, it must be borne in mind, that the figure by which the proof is commonly accompanied is not itself the subject of reasoning, but merely an illustration in order to aid the student in conceiving the species to which the reasoning relates, and to enable the

geometer to speak with clearness of its separate parts. So long as it serves this purpose, it matters little how rude the illustration may be. On the other hand, the possibility of a construction properly framed will need no extraneous proof. The student of geometry must obviously have credit for the conception of the figures which form the subject of the science; and to that effect he must be acquainted with the elementary materials of form, with the attributes and relations by which they are to be moulded into definite systems. Now, the impossibility of a certain construction, or its incapacity of actual existence, must arise from some essential incongruity in the conditions on which the construction is based, and the same objection would equally be fatal to the distinct conception of the system. All our ideas being ultimately derived from experience, whatever can be distinctly conceived is inherently capable of exhibition in actual existence. The student, therefore, will carry in his own mind the only proof he requires of the possibility of any construction which he can distinctly imagine, nor will his conviction in such possibility be in any way increased by a problem shewing him a particular means of mechanical execution. Thus the capacity of division into parts being an essential attribute of every kind of magnitude, the student will be capable of conceiving the division of any magnitude, as an

angle, into two equal parts, or in any other proportion, and will require no further proof of the possibility of such an operation when called for in the course of demonstration. The geometer will, of course, have to take care that he proposes such constructions only as the student is able to carry out, an end which will effectually be secured so long as the position of each fresh element to be used in construction be defined by a simple relation to portions already fixed in the system.

The principle of arrangement adopted in the following system, has been to place the propositions in an order in which each admits of being proved by direct reasoning, wholly avoiding the employment of *ex absurdo* demonstration. For that purpose, it has been found necessary to take some propositions relating to circles before all the geometry of triangles has been exhausted; and thus, somewhat to disturb the symmetry of Euclid's arrangement. It is not to be supposed that the conclusion from direct, is of greater cogency than that from *ex absurdo* reasoning. But, in the one case, the reasoning shews that the subject of the proposition *is* in the demonstrated predicament; in the other, that it *must* be so, leaving a craving in the mind to know how the necessity directly arises out of the essential nature of the figure. The two modes of proof are, in fact, examples of what are called synthetical and analytical reason-

ing respectively. In the former, the reasoner starts from elements completely known in the aspect under which they present themselves to the faculties of actual apprehension, and by combining them in known relations, he attains the conception of more complex objects known in like manner, the complex relations of which it is the office of the reasoning to show from the principle of their construction.

In analytical reasoning, on the other hand, the subject of reasoning is, in the first instance, known only from some relation to other objects, or complex relation between its own parts; and the object of reasoning is to determine the fundamental nature of the subject, or the features by which it is known in the act of simple apprehension. Now, in *ex absurdo* demonstration, the real object of research is to ascertain whether a given relation between certain elements is or is not in accordance with the fundamental constitution of the system. For the purpose of trying this question, the affirmative is assumed; a secondary system is constructed on the basis of the assumed relations, the attributes of which the geometer proceeds to investigate, until he arrives at some conclusion directly opposed to his previous knowledge. He concludes that the assumption, of which the absurd conclusion is a necessary consequence, must itself be untrue, or, in other words,

that the relation under inquiry is incompatible with the nature of the elements between which it was supposed to hold good. It is plain, then, that the system constructed on the erroneous supposition cannot be distinctly conceived or completely known as an object of actual apprehension; and the reasoning by which the essential incongruity of the construction is brought to light will fall under the head of analytical reasoning. Thus, all *ex absurdo* demonstration should, if possible, be excluded from geometry, the peculiar characteristic of which is supposed to be, that it affords a perfect example of synthetical reasoning.

DEFINITIONS.

I.—Position.

The position of a point is the relation depending on the character, in respect of distance and direction, of the space by which it is separated from some fundamental point; so that points connected with the same fundamental point by tracks identical, in respect of distance and direction, are in the same position; and points in the same position are at the same distance in any given direction from the same fundamental point.

II.—Opposition.

Two directions are opposed to each other when distance in the one, in the determination of position, exactly nullifies the same amount of distance in the other; that is to say, when the spectator having advanced a certain distance in the one direction, is brought back by the same distance in the other to the position originally occupied.

III.—Transverseness.

A direction is transverse to another when motion to any extent in the second is compatible with a total absence of motion in the first; or when motion in the second is wholly without effect in producing motion in the first.

IV.—Obliquity.

A direction is oblique to another when distance in the one essentially constitutes distance in the other as well as in a direction transverse to it, or when distance in the one admits of being wholly decomposed into distance in the other, and in a direction transverse to it.

V.—Rigidity or Fixedness.

A rigid or fixed system is one in which all the points remain in the same relative position, while the entire system may be moved from place to place in external space.

VI.—Coincidence.

Points in the same position are said to coincide. Lines, surfaces, and solids coincide respectively when every point in each of the magnitudes compared coincides with a corresponding point in the other.

VII.—Straight Line.

A straight line is a line passing through each successive point that can be reached from a given point by motion in a single constant direction.

VIII.—Parallel Straight Lines.

Parallel straight lines are straight lines lying in the same direction in a system and not forming parts of the same straight line.

IX.—Plane Surface.

A plane is a surface passing through every point that can be reached from a given point under the condition of a total absence of motion in a certain constant direction, called the *normal* to the plane, or, in other words, through every point which can be reached by motion transverse to a certain constant direction.

X.—Circle.

A circle is a line passing through every point in a plane which lies at a certain given distance from a point called the *centre of the circle.*

A straight line from the centre to the circumference is called the *radius.*

XI.—Triangle.

A triangle is a plane rectilinear figure of three sides.

XII.—Parallelogram, Rectangle, Square.

A parallelogram is a plane rectilineal figure of four sides of which the opposite sides are parallel to each other.

When the angles are all right angles the parallelogram becomes a rectangle; and when all the sides are equal the figure is a square.

XIII.—Equal, Greater, Less.

Magnitudes are equal to each other when the one may be made exactly to coincide with the other, or when they may be divided into a number of parts, so that the sum of all the parts of the one may be made to coincide with the sum of all the parts of the other.

But when the magnitudes are such that the whole of the one may be made to coincide with a part of the other, or when the sum of all the parts of the one is contained within the sum of all the parts of the other, leaving some part of the latter unoccupied, the containing magnitude is the greater, the contained the less.

XIV.—Angle.

Two straight lines meeting in a point are said to make an angle at that point, the magnitude of which is measured by the quantity of plane surface abutting on the point of intersection between the straight lines or arms of the angle.

XV.—Right Angle.

When a straight line standing on another straight line makes the adjacent angles equal to each other, each of the angles is called a *right angle*, and the straight line which stands on the other is called *perpendicular*.

PROPOSITIONS.

I.

Straight lines from the same point and in the same direction coincide.

Let A B, A C, be two straight lines lying in the same direction from the point A. In either of these as A B take any point b, and let c be a point in A C, such that A c is equal to A b. Then the tracks, by which the position of the points b and c are determined, are identical in respect of distance and direction, and the points b and c coincide; that is to say, any point in either of the lines A B, A C (and, therefore, every point in each of those lines) corresponds with a corresponding point in the other. Therefore, the lines A B, A C, wholly coincide.

II.

If two straight lines coincide in any two points, they coincide throughout to the extent of their joint length.

Let the straight lines 1 and 2 coincide in the points A and B, and let C be any point in line 1, on the same side of A with the point B; C′ a poin in line 2, on the same side of A with B, and at the same distance as C from A.

Then, because A, B, and C, are points in the same straight line, B and C lie in the same direction from A; and, for a like reason, the points B and C′ are in the same direction from A. Therefore the points C and C′, being in the same direction from A, with a common point B, are in the same direction from that point with each other; and they are, by the construction, at the same distance from A, therefore the points C and C′ coincide. That is to say, any point in either of the lines 1 and 2 (as far as they jointly extend in distance) coincides with a corresponding point in the other, and therefore the lines coincide throughout.

III.

Straight lines joining any two points in a plane surface fall wholly within the plane.

Let A B be a straight line joining any two points in a plane. Then the spectator, moving along the plane from A to B, will be without motion in the direction of the normal (Def. 9); or in other words, the distance of B from A in the direction of the normal is null. Therefore, a spectator moving direct from A to B, along the line A B, will be without motion in the direction of the normal; and A B, by the definition, will be a line in the plane.

IV.

Straight lines in opposite directions from the same point form parts of a single straight line.

Let A B, A C, be straight lines in opposite directions from the point A, and let the spectator proceed along A B from A to B. Then B A is manifestly the direction in which he must return, in order to reach the point A from whence he started; that is to say, B A is the opposite to A B, and is therefore in the same direction with A C, and B A C forms a single straight line.

V.

All right angles are equal.

Let B A C, E D F, be any two right angles (fig. 2), and let B A and E D be produced to G and H respectively.

Let the plane E D F H be superimposed upon the plane B A C G, so that the point A shall coincide with the point D, and the straight line E D H with B A G. Then the plane surface abutting on the point D, between D E and D H, on the side towards F, consisting of the angular spaces E D H, F D H, will coincide with the surfaee abutting on the point A, between A B and A G, on the side towards C, consisting of the angular spaces B A C, C A G. Therefore the angles E D F, F D H, are together equal to the angles B A C, C A G. But

the two together, E D F, F D H, are together double of E D F; and B A C, C A G, are double of B A C. Therefore B A C is equal to E D F.

Cor. 1.—As division produces no alteration in the magnitude of a figure, the angular expansion round a point in a straight line on one side of the line is equal to two right angles, in whatever manner it may be divided, whether into two equal or unequal angles, or into any number of angles.

Cor. 2.—The whole angular expansion round a point in a plane is equal to four right angles.

VI.

If a straight line standing on two other straight lines make the adjacent angles equal to two right angles, the two straight lines form a single straight line.

Let B D (fig. 3) be a straight line, making with the straight lines B A, B E, the angles D B A, D B E, equal to two right angles; the straight lines B A, B E, will form a single straight line A B E.

Let A B be produced on the other side of B to any point F. Then the angular expansion A B D, D B F, is equal to two right angles, and therefore, by the hypothesis, to the two angles D B A, DBE. Take away the angle A B D, and the remaining angle D B F will be equal to the remaining angle

D B E. Therefore the arm B F coincides with B E, and A B E is a straight line.

VII.—Euclid I. 15.

The vertical or opposite angles, made by two straight lines cutting each other, are equal to each other.

Let the two straight lines A B, C D (Fig. 4), cut one another in the point E, the angle A E C shall be equal to the angle D E B, and C E B to A E D.

The angles A E C and C E B, on one side of the straight line A B, are together equal to two right angles; and the angles C E B, B E D, on one side of the straight line C D, are also equal to two right angles; wherefore the angles A E C, C E B, are together equal to the two C E B, B E D. Take away the common angle C E B, and the remaining angle C E A is equal to the remaining angle D E B.

In the same way it may be proved that the angles C E B, A E D, are equal.

Cor.—The four angles made by two straight lines crossing each other at a right angle, are all right angles.

VIII.

The perpendicular is transverse to the straight line on which it stands, and conversely a straight line meeting another in a transverse direction is at right angles to it.

Let C D (Fig. 5), be a straight line perpendicular to C A, and let C A be produced on the other side of C, to a point B, and let C A point to the left, C B to the right. Now let the point C remain fixed while the plane C A D B is turned over upon its face, so that the position of every point in the system shall be precisely reversed with respect to right and left of the point C, while no difference is made in the position of points in a direction up and down the paper; and let A′ B′ D′ be the position of A, B, D in the reversed system. Then as C B formerly pointed directly to the right, it will now point left, and thus, as C B′, will coincide with C A, and C A′ for the same reason with C B. But the angle B C D equals the angle A C D, being both right angles; therefore the angle B′ C D′ (which is the angle B C D in its new position) or A C D′ is equal to the angle A C D. Therefore C D′ coincides with C D, or, in other words, there is no change in the position of C D. No point, therefore, in C D, can lie either to the right or left of the point C, or the spectator, advancing along C D, would be without motion in the direction C A or C B; that is to say, C D is transverse to C A and C B.

Next, let C D be transverse to C A, C B. Then let the position of the plane C A D B be reversed as before, and A′, B′, be the new positions of the points A, B.

Then, because C D is transverse to C B, no point

in C D lies either to the right or left of C, and the position of C D will remain unaltered in the reversed figure. Moreover, C B′, for the same reason as before, will coincide with C A, and the angle B′ C D with the angle A C D; therefore, the angle B C D is equal to A C D, or they are both right angles, and C D is perpendicular to C A or C B.

Cor. If one direction be transverse to a second, the second is also transverse to the first.

IX.

If two straight lines pointing in the same direction make equal angles with two other straight lines in the same plane, the latter are also in the same direction.

And conversely if two straight lines in the same direction with each other intersect two other straight lines also in the same direction with each other, the angles between each pair of intersecting lines are equal.

If A B, D E (Fig. 6) be straight lines in the same direction and they meet the lines A C, D F in the same plane, making the angle B A C equal to E D F, the straight line A C shall be in the same direction with D F.

And conversely, if A B and A C be in the same direction with D E, D F respectively, the angle B A C shall be equal to the angle E D F.

First, let the angle B A C be equal to the angle

E D F; join A D, and let the angle B A C slide along the line A D without any change of direction in any of the lines of the system until the point A is brought to coincide with D. Then, because D E is in the same direction with A B, the two lines will coincide; and because the angle B A C is equal to E D F, when A is brought to D the angles will also coincide, and the line A C will coincide with D F; therefore A C is in the same direction with D F.

In the next place, let A B, A C be in the same directions respectively with D E, D F; and the same construction being made when A is brought to D, A B will coincide with D E, and A E with D F; therefore the angle B A C is equal to the angle E D F.

X.

Two parallel lines may always be included in the same plane.

Let A B, C D (Fig. 7) be parallel straight lines, A C a straight line cutting them, and let A E, C F be straight lines in the direction of the normal to the plane A C, A B.

Then, because C D is in same direction with A B, it will be transverse to every direction to which A B is transverse, and, therefore, to the direction A E or C F. C D is, therefore, a line in the same plane with A B and A C.

XI.

Parallel straight lines make equal angles with a straight line cutting them; and, conversely, straight lines in the same plane making equal angles with a straight line cutting them are parallel to each other.

Let A B, C D (Fig. 7) be parallel straight lines, A C G a straight line cutting them in A and C respectively; the angle G A B shall be equal to G C D.

Because A C G is a straight line, A C and C G are in the same direction; and A B and C D, being parallel, are also in the same direction with each other. Wherefore (Prop. 9) the angle C A B is equal to G C D.

Next, let A B, C D, be straight lines in the same plane, making the angle C A B equal to G C D; C D shall be parallel to A B.

Because A C is in the same direction with C G, and the angle C A B is equal to G C D, therefore (Prop. 9, case 2) C D is in the same direction with A B, that is, C D and A B are parallel.

XII.

Parallel straight lines do not meet if produced ever so far both ways.

Let any straight lines A E. C E (Fig. 4) meet in the point E, and let them be produced beyond E to B and D; they cannot again meet if produced ever so far (Prop. 2). Then C E and A E are in the directions E D and E B respectively,

that is to say, they are in different directions, and are, therefore, not parallel; that is to say, lines which meet at any distance however great, are not parallel, and therefore, conversely, parallel lines never meet.

XIII.

If a straight line fall on two parallel lines, it makes the alternate angles equal to each other, and the two interior angles equal to two right angles.

Let F B D, C E be parallel straight lines (Fig. 8); A B C a straight line cutting them in B and C respectively; the angle F B C shall be equal to B C E, and the angles D B C and D C B shall be together equal to two right angles.

Because B D and C E are parallel, the angle A C E is equal to the angle A B D, and, therefore, to the vertical angle F B C.

To each of these add the angle D B C. Then the angles A C E, D B C are together equal to the two F B C, D B C, that is. to two right angles.

XIV.—Euclid I. 32.

If one side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three angles of every triangle are together equal to two right angles.

Let A B C (Fig. 9) be a triangle, and let one of its sides, B C, be produced to D; the exterior

angle A C D is equal to the two interior and opposite angles C A B, A B C, and the three interior angles A B C, B C A, C A B, are together equal to two right angles.

Through the point C draw C E parallel to A B; and because A B is parallel to C E, and A C meets them, the alternate angles B A C, A C E are equal. Again, because A B is parallel to C E, and B D falls upon them, the exterior angle E C D is equal to the interior and opposite angle A B C. Therefore the whole angle A C D, consisting of the two angles A C E, E C D, is equal to the two interior and opposite angles A B C, B A C.

To each of these add the angle A C B. Then the three interior angles of the triangle A B C, B A C, B C A will be together equal to the two, A C B, A C D; that is, to two right angles.

Cor. If from any point in a straight line making an oblique angle with a second line a third straight line be drawn perpendicular to the second, the perpendicular will fall on the same side of the oblique line with the acute angle, and on the opposite side to the obtuse angle.

XV.—Euclid I. 32, *Cor.*

All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For any rectilinear figure A B C D E (Fig. 10)

can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of the angles. And by the last proposition all the angles of these triangles are equal to twice as many right angles as there are triangles or sides to the figure. And the same angles are equal to the internal angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is, together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

XVI.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles equal, the third sides are also equal, and the remaining angles of the one to the remaining angles of the other, viz. those to which the equal sides are opposite.

Let A B C, D E F (Fig. 11) be two triangles, having the sides A B, A C, equal to D E, D F, respectively, and the angle B A C equal to E D F. The third side B C will also be equal to E F, and the angles A B C, A C B to the angles D E F, D F E respectively.

Let the triangle A B C be superimposed on D E F, so that the point A shall lie on D, A B

upon D E, and the plane A B C upon the plane D E F.

Then, because A B is equal to D E, the point B will coincide with E; and because the angle BAC is equal to E D F, and A B coincides with D E, the line B C will lie on the line D F; and because A C is equal to D F, point C will coincide with D, and therefore line D C with D F, and the triangles will wholly coincide. Therefore the angle A B C is equal to D E F, and angle A C B to D F E.

Cor.—In triangles having two sides of the one equal to two sides of the other respectively, if the bases are not equal, the angles at the vertex are unequal.

XVII.

If two triangles have two angles and one side of the one equal to two angles and the corresponding side of the other, the triangles are altogether equal; viz. the remaining sides of the one to the remaining sides of the other to which the equal angles are opposite.

If two angles of the one triangle are equal to two angles of the other, the third angles are also equal (Prop. 14).

Let the triangles A B C, D E F (Fig. 11), have the angles B A C, B C A, equal to E D F, E F D, respectively, and therefore the third angle A B C equal to D E F, and let any side A C be equal to

D F. The remaining sides A B, B C shall also be equal to E D, E F respectively.

Let the triangle A B C be superimposed on D E F, so that A C shall coincide with D F. Then, because the angle B A C is equal to E D F, the line A B will lie on the line D E, and for a like reason the line C B will lie on F E. And because B is a point in line A B, it will fall somewhere in line D E, and because it is a point in line C B it will fall somewhere in F E. It will therefore fall on the point E in which D E and F E intersect each other, and the sides A B, B C will coincide with the sides D E, F E respectively. Therefore the sides A B, B C are equal to D E, F E, respectively, and the triangles are equal in every respect.

XVIII.

The angles at the base of an isosceles triangle, or a triangle having two equal sides, are equal to each other; and, conversely, if the angles at the base of a triangle are equal, the opposite sides are also equal to each other.

Let A B C (Fig. 12) be a triangle in which A B is equal to A C. The angle A B C will be equal to A C B.

Let the triangle A B C be taken up and laid on its face, so that the side A B which was on the left of the triangle shall now be on the right, and let A′ C′ B′ be the triangle in the reversed position.

Then A B C, A′ C′ B′ will be two triangles, having the sides A B, A C, equal to the sides A′ C′, A′ B′, respectively, and the angle B A C equal to C′ A′ B′. Therefore the remaining angles of the one are equal to the remaining angles of the other; that is to say, the angle A B C is equal to A′ C′ B′, and therefore to the angle A C B.

Next, let the angle A B C equal A C B. The side A B shall be equal to A C.

The same construction being made, the triangles A B C, A′ C′ B′, will be triangles having the angles A B C, A C B, respectively equal to angles A′ C′ B′, A′ B′ C′, and the side B C equal to C′ B′. Therefore the remaining sides are equal, and A B is equal to A′ C′ or to A C.

Cor.—If the angles at the base of a triangle are not equal, the opposite sides are unequal, and if the sides of a triangle are not equal the opposite angles are unequal.

XIX.—Euclid I. 18.

If one side of a triangle be greater than another, the angle opposite the greater side is greater than the angle opposite the less.

Let A B C (Fig. 13) be a triangle of which the side A C is greater than A B; the angle A B C shall be greater than A C B.

Because A C is greater than A B, let A D, part of A C, be equal to A B; join B D. Then because

A D B is the exterior angle of the triangle B D C, it is greater than the interior and opposite angle D C B. But A D B is equal to A B D, because A B D is an isosceles triangle. Therefore the angle A B D is likewise greater than the angle A C B. Much more then is the angle A B C (which consists of A B D and D B C) greater than A C B.

XX.

If one angle of a triangle be greater than another, the side opposite the greater angle is greater than the side opposite the less.

Let A B C be a triangle, of which the angle A B C is greater than A C B; the side A C will be greater than A B.

Because the angles at the base of the triangle A B C are not equal, the opposite sides are unequal (Prop. 18, *Cor.*); that is, one of the two A B, A C, is greater than the other. But in a triangle having one side greater than another, the angle opposite the greater side is greater than the angle opposite the less; and in the triangle A B C the angle A B C is greater than A C B; therefore the angle A B C is opposite the greater of the two A B, A C, or, in other words, A C is greater than A B.

XXI.—Euclid I. 20.

Any two sides of a triangle are together greater than the third side.

Let A B C be a triangle (Fig. 14), take any side B A and produce it at one extremity to D, making A D equal to A C the adjacent side, and join D C.

Because A D is equal to A C, the angle A D C is equal to A C D, and the angle B C D is equal to B C A and A C D, and is therefore greater than A C D, or A D C; and because the angle B D C is a triangle in which the angle B C D is greater than B D C, therefore the side D B opposite B C D is greater than the side B C opposite B D C. But D B is equal to B A and A D; that is, to B A and A C. Therefore B A and A C, or any two adjacent sides, are greater than the third side B C.

XXII.—Euclid I. 21.

If from the ends of the sides of a triangle two straight lines be drawn to a point within the triangle, these shall be less than the other sides of the triangle, but shall include a greater angle.

Let the two straight lines B D, C D (Fig. 15) be drawn from B, C, the ends of the side B C of the triangle A B C, to a point D within it; B D and C D together shall be less than the other sides B A, A C of the triangle, but shall contain an angle B D C greater than the angle B A C.

Produce B D to meet the side of the triangle in E. Then the two sides B A, A E of the triangle A B E are greater than the third side B E. To each of these add E C. Therefore the sides B A, A C are

greater than B E, E C. Again, because the two sides C E, E D of the triangle C E D are greater than C D, to each of these add D B; therefore the sides C E, E B are greater than C D, D B. Much more then are the sides B A, A C greater than B D, D C.

Again, because the exterior angle of a triangle is greater than the interior and opposite angle, the exterior angle B D C of the triangle C D E is greater than C E D; for a like reason, the angle B E C is greater than B A C. Much more then is the angle B D C greater than B A C.

XXIII.—Euclid I. 24.

If two triangles have two sides of the one equal to two sides of the other each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let A B C, D E F (Fig. 16) be two triangles, having the two sides A B, A C equal to the two D E, D F each to each; and the angle B A C greater than the angle E D F; the base B C shall also be greater than the base E F.

Of the sides D E, D F let D E be the side which is not greater than the other, and let D G be a straight line making the angle E D G equal to

B A C ; let D G be equal to A C or D F ; G F, G E straight lines joining G with F and E.

Then, because A B is equal to D E, and A C to D G, and the angle B A C to E D G, therefore the base B C is equal to E G; and because D G is equal to D F, the angle D F G is equal to D G F, but the angle D G F is greater than the angle E G F, and much more is the angle E F G greater than the angle E G F ; therefore the side E G of the triangle E G F, which is opposite the angle E F G, is greater than E F, the side opposite E F G; but E G is equal to B C, therefore B C is greater than E F.

XXIV.

If two triangles have two sides of the one respectively equal to two sides of the other, but the base of the one greater than the base of the other, the angle also contained by the sides of that which has the greater base is greater than the angle contained by the sides equal to them of the other triangle.

In the triangles A B C, D E F (Fig. 17) let A B, A C be equal to D E, D F respectively, and let E F be greater than B C. Then the angle E D F shall be greater than the angle B A C.

Because the bases B C, E F are not equal (by Prop. 16, *Cor.*), the opposite angles B A C, E D F are unequal, or one of them is the greater of the

two and (by Prop. 24) the base opposite the greater angle is the greater. But in triangles A B C, D E F the base E F is greater than A C; therefore E F is opposite the greater of the two angles B A C, E D F; that is to say, the angle E D F, to which E F is opposite, is greater than B A C.

XXV.

If from a point without a straight line a number of straight lines be drawn to the former one whereof one is perpendicular, the shortest line is the perpendicular, and of the others the one making a smaller angle with the perpendicular is shorter than one making a larger angle with it; and only two equal straight lines can be drawn from the point to the straight line, viz., at equal angles on either side of the perpendicular.

Let A (Fig. 18) be a point without the straight line D B; A B perpendicular to D B; A C a straight line from A cutting D B in C; A D another similar line, making angle D A B greater than C A B; A E falling on D B on the other side of B, making angle B A E equal to B A C.

Then A B will be the shortest of all the lines A B, A C, A D, etc.; and of the others, A C is shorter than A D, and A E is the only line that can be drawn from A to D B equal to A C.

Because the angle A B C is a right angle, the angle A C B is less than a right angle; wherefore

the side A C is greater than A B, or A B is shorter than any other line drawn from A to D B.

Again, because the angle A C B is less than a right angle, the angle A C D in the triangle A C D is greater than a right angle, and, therefore, the angle A D C is less than a right angle, or the angle A C D of the triangle A C D is greater than A D C; wherefore the side A D is greater than the side A C.

And because the angle B A E is equal to the angle B A C, and the angle A B E is equal to the angle A B C (being both right angles), the triangles A B C, A B E have two angles of the one equal to two angles of the other, and the side A B common to the two. Therefore the triangles A B C, A B E are equal, and the side A E is equal to A C, and every other line on the other side of B from C is (by the present Prop.) either greater than A E or less; wherefore A E is the only line that can be drawn from A to D B equal to A C.

Cor. 1.—If a straight line be shorter than any other that can be drawn from a point without a straight line to that line, the former straight line is perpendicular to the latter.

Cor. 2.—The perpendicular on the base of an isosceles triangle bisects the angle at the vertex.

XXVI.

The straight line joining any two points in the

circumference of a circle falls wholly within the circle.

Let A B (Fig. 19) be a straight line joining A and B points in the circumference of a circle whose centre is C. A B will fall wholly within the circle.

Let C A, C B be radii of the circle at the points A and B; C D perpendicular to A B. Then (Prop. 25, *Cor.* 2) C D bisects the angle B A C, and every straight line from C to the straight line A B, between C A and C B, will be less than C A or C B (Prop. 25), and therefore every point in A B falls within the circle.

XXVII.

If a straight line joining any two points in the circumference of a circle be bisected, and a perpendicular erected at the point of bisection, the centre of the circle lies in the perpendicular.

Let A and B (Fig. 20) be points in a circle, and let the straight line A B be bisected in D; D C perpendicular to A B. The centre of the circle will be in the line D C.

Let F be any point within the circle not in the straight line D C. Join F A, F B, of which F A cuts D C in E. Join E B. Then E A is equal to E B (Prop. 26), and A F is equal to A E and E F, that is, to E B and E F. But E B and E F are together greater than F B, the third side of the triangle E B F; therefore A F is greater than F B.

Wherefore F is not the centre of the circle, or, in other words, the centre of the circle does not lie in any point without the perpendicular C D, that is, it lies upon it.

XXVIII.

If a straight line cut a circle, it shall cut it again at a point at an equal distance on the other side of the perpendicular from the centre and at no other point.

Let the straight line A B (Fig. 19) cut the circle whose centre is C, at the point A, and let C D be perpendicular upon A B. A B shall cut the circle again at a point B so that D B is equal to D A, and at no other point.

Join C B, C A. Then because D A is equal to D B, and C D is common to the triangles C D A, C D B, and the angles C D A, C D B are both right angles, the sides D A, C D are respectively equal to the two D B, D C, and the included angle C D A is equal to the angle C D B; wherefore the triangles are equal and the base C B is equal to C A, or B is a point in the circle, and the line A B (Prop. 27) falls wholly within the circle.

And because no other line besides C B can be drawn from C to A B equal to C A (Prop. 26), no other point in A B coincides with any point in the circle; or, in other words, the straight line A B cuts the circle in no other point.

XXIX.

The perpendicular at the extremity of a diameter falls without the circle, and no straight line can be drawn between that and the circle so as not to cut the circle.

Let A B (Fig. 21) be the diameter of a circle; C, the centre; B E perpendicular to B C. B E will fall wholly without the circle, and no straight line can be drawn between B E and the circle so as not to cut the circle.

Let C F be a line joining C and any point F in B E. Then, because C B F is a right angle, the angle C F B is less than a right angle; wherefore C F is greater than C B, or the point F lies without the circle.

Next let B D be any other straight line through B, and let C D be perpendicular to B D. Then because C D B is a right angle, C D B is greater than C B D, and C B is greater than C D; or the point D falls within the circle. And any line from C to D B, beyond C B, will be greater than C B; wherefore every part of the line D B on the other side of B will fall without the circle. Therefore the circle intersects D B in B.

Cor.—If a straight line touch a circle, the centre of the circle lies in the perpendicular to the touching line at the point of contact.

XXX.—Euclid III. 7.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines that can be drawn from it to the circumference the greatest is that in which the centre is, and the other part of the diameter is the least; and of any others, that which makes a less angle with the line which passes through the centre is greater than the one which makes a greater angle: and from the same point there can only be drawn two straight lines that are equal to one another, one upon each side of the shortest line.

Let A B C D (Fig. 22) be a circle, and A D its diameter in which let any point F be taken which is not the centre. Let the centre be E; of all straight lines drawn from F to the circumference, F A, F B, F C, etc., F A is the greatest and F D is the least: and of the others, F B which makes a smaller angle with F A is greater than F C which makes a greater angle with it.

Join B E, C E, and because two sides of a triangle are greater than the third, B E, E F are greater than B F. But A E is equal to E B, therefore A E, E F, that is A F, is greater than B F.

Again, because B E is equal to C E, the two sides B E, E F are equal to the two C E, E F, but the angle B E F is greater than C E F; therefore the base B F is greater than the base C F. Again,

because C F, F E are greater than E C, and E C is equal to E D; C F, F E are greater than E D, that is, than E F, F D. Take away the common part F E, and the remainder C F is greater than F D. Therefore F A is the greatest and F D the least of all the lines from F to the circumference.

Also there can only be drawn two equal lines from F to the circumference, one upon each side of the shortest line F D. At the point E let the line E F make the angle F E H equal to the angle C E F and join F H. Then because C E is equal to E H, and F E is common to the triangles F E H, C E F, the two sides C E, F E are equal to the two F E, E H respectively, and the angle C E F is equal to the angle H E F; therefore the base F C is equal to the base F H. And every other line from F to the circumference beyond F H is, by the proposition, greater than F H, and any line between F H and F D is smaller than F H, therefore there is no other line than F H equal to F C.

XXXI.

If any point be taken without a circle and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference the greatest is that which passes through the centre, and of the rest, that which makes a less angle with the line passing through the centre is greater

than that which makes a greater angle with it. But of those which fall upon the convex circumference, the least is that, which produced, passes through the centre; and of the rest, that which makes a less angle with the shortest line is less than that which makes a larger angle with it. And if a line be drawn touching the circle, it shall be the greatest of all the lines falling on the convex circumference and less than any of those falling on the concave circumference. And only two equal straight lines can be drawn from the point to the circumference, one upon each side of the shortest line.

Let D (Fig. 23) be a point without a circle G A F; D A, D E, D F (of which D A passes through M the centre of the circle) straight lines from D to the concave part of the circumference, falling on the convex part in G, K, L, respectively. Then of the lines D A, D E, D F, etc., D A is the greatest; and of the others, D E, which makes a smaller angle with D A, is greater than D F, which makes a larger angle with it; and of the lines D G, D K, D L, etc., D G is the shortest; and of the others, D K, which makes a smaller angle with D G, is less than D L, which makes a greater.

Join M E, M F, M K, M L. Then because A M is equal to E M, add M D to each, therefore A D is equal to E M, M D. But E M, M D are greater than E D; therefore A D is greater than E D.

And because the two D K, M K are greater than M D, or D G, G M, and M G is equal to M K, the remaining linc D K is greater than D G, or of lines falling on the concave circumference, the largest is that which passes through the centre; and of those which fall on the convex circumference, the shortest is that which, produced, passes through the centre.

Again, because the line D E cuts the circle in K and E, the part of the circle on the other side of D E, lies between the straight lines M K, M E; wherefore if a line on the other side of D E cuts the circlc at all, it will cut it at points situate between M E and M K; therefore thc lines M F and M L lie on the other side of M E and M K from M A and M D respectively, or the anglc D M E is greater than the angle D M F, and D M L is greater than D M K.

But M F is equal to M E, and D M is common to the triangles D M E, D M F; therefore the base D E is greater than the base D F, or of lines falling on the concave circumference, the one making a less angle with D M is greater than onc making a greater angle.

Again, because M L is equal to M K, and M D common to the triangles D M L, D M K, but the angle D M L greater than D M K, the base D L is greater than the base D K; or of lines falling on the convex circumference, the one

making a greater angle with D M is greater than the one making a less.

Next, let D P be drawn touching the circle at P;. D P shall be the greatest of all the lines D G, D K, D L, etc., and less than any of the lines D F, D E, etc.

Because D P touches the circle at P, no line on the other side of D P can touch the circle, or M D P is the greatest angle at which a line can be drawn from D so as to meet the circle; and, therefore (by the present Prop.), D P is the greatest of all the lines falling on the convex circumference.

Join M P: and because D P is the furthest line from D A which meets the circle, every other line on the same side of D A, which meets the circle, lies between D P and D A, and therefore cuts the line P M somewhere between P and M. Let D F be any such line cutting P M in N, and falling on the concave circumference at F.

Then because D P N is a right angle (Prop. 30, *Cor.* 1), the angle D P N is greater than D N P, and D N is greater than D P. Much more, then, are D N and N F or D F greater than D P, or the line touching the circle is less than any of the lines falling on the concave circumference.

Also, there can be but one line drawn from D to the circumference equal to D K, viz., on the other side of the least line D G.

Let the angle D M B be equal to the angle

D M K. Join D B. Then because M B is equal to M K, M D common to the triangles D M B, D M K, and the angle D M B equal to D M K, the base D B is equal to D K. And every line drawn to meet the circle between D G and D B is less than D B, and on the other side of D B is greater than D B. Therefore, D B is the only line which can be drawn from D to the circle equal to D K.

XXXII.

If two circles touch each other internally or externally, the straight line joining their centres passes through the point of contact.

First, let two circles touch each other internally at the point A (Fig. 24), and let A F be the straight line touching the external circle; D the centre of the external, C of the internal circle. The straight line D C produced will pass through A.

Join A D: and because A F touches the circle D in point A (Prop. 29, *Cor.*) A D is perpendicular to A F, and because the circle C meets the circle D at the point A, and the circle D meets the straight line A F at the same point, therefore the circle C also meets the straight line A F at the point A. But the circle D is external to the circle C, and the line A F is external to the circle D; much more then is the line A F external to the circle C. Therefore the line A F touches the circle C at the point A; and because A D is perpendicular to A F (Prop. 29,

Cor.) the centre C lies in the line A D, and the straight line D C prolonged passes through A, the point of contact.

Next, let the circles whose centres are C and D touch each other externally at the point A (Fig. 25). Join DA, C A, and let D G be a straight line from the centre D of either of the circles to the circumference of the other. Then, because the circles touch each other externally at the point A, the shortest line that can be drawn from D to the circle C cannot be less than D A, or, in other words, D A cannot be greater than any other line drawn from D to the circle C. But every line from D to circle C not passing through the centre (Prop. 32) of the latter circle is greater than the line passing through the centre. Therefore D A cannot be one of the lines from D to the circle C not passing through the centre C, that is, it does pass through that point.

Cor.—One circle can touch another at one point only, whether internally or externally.

XXXIII.

If two circles intersect each other, they intersect each other twice, at equal distances on either side of the straight line joining their centres, and at no other point.

Let A and B (Figs. 26 and 27) be two circles intersecting each other at the point D; C and K

their centres; the two circles shall also intersect each other at a point E, as far below the line C K as D is above it, and at no other point.

Join either of the centres C with D, and let D E be perpendicular to C K (produced if necessary) cutting it at F, and let F E be equal to F D. Then because C F is perpendicular to D E (Prop. 25) every straight line from C to any point in D E between D and F, is less than C D, or is within the circle, and every line to any point in D E, beyond D, is greater than C D, or is without the circle; wherefore D E cuts the circle A in D; and because C F is perpendicular to B F, and F E is equal to D F (Prop. 28), therefore D E cuts the circle A again at the point E; and, for a like reason, D E intersects the circle B at the same point. Therefore the circles A and B meet at the point E, and as the point E is not in the line joining the centres of the circles, they cannot touch each other at that point, therefore they must intersect each other.

Nor can the circles intersect at any other point. Two cases may be distinguished according as the centre of one of the circles is without the other circle or not without it.

First, let the centre C of one of the circles A be without the other circle (Fig. 26). Then, because C is a point without the circle B, and C K a line passing through the centre, and C D and C E equidistant on either side of C K (Prop. 30), every line

from C to circle B, between C D and C E, will be less than C D or C E; and, therefore, the segment of the circle B, between C D and C E, will fall within the circle A. Moreover, every line beyond C D on the one side, and C E on the other, falling on the convex circumference of the circle B, and every line falling on the concave circumference, will be greater than C D or C E, and will, therefore, fall without the circle A. Thus, the circles will only meet in the points D and E.

Next, let neither of the centres fall without the other circle (Fig. 27). Then, because C is a point in the diameter of circle B which is not the centre (Prop. 31), every straight line from C to the circumference of circle B, between C D and C E, will be less than C D or C E, and the segment D E of the circle B will fall wholly within the circle A. And every straight line from C to the remaining circumference will be greater than C D or C E, and will, therefore, fall without the circle A, and the two circles will meet each other in the points D and E only.

XXXIV.

If two triangles have three sides of the one equal to three sides of the other, each to each, the angles of the one shall be equal to the angles of the other each to each, namely, those to which the equal sides are opposite.

Let A B C, D E F (Fig. 11) be two triangles having the three sides A B, B C, C A, respectively equal to the three D E, E F, F D. The angles B A C, B C A, A B C shall be equal to the angles E D F, E F D, D E F respectively.

Let the triangle A B C be superimposed on the triangle D E F, so that the line A C shall lie on the line D F. Then, because B is a point in the line A B, and A B is equal to D E, the point B will be found somewhere in the circumference of a circle whose centre is D and radius D E; and because it is also a point in the line B C, and B C is equal to E F, the point B will also be found somewhere in the circumference of a circle whose centre is F and radius E F. It will, therefore, be found in the only point E in which the circles intersect above the line of centres, and the two triangles will wholly coincide; and, therefore, the angles of the one are equal respectively to the angles of the other to which the equal sides are opposite.

XXXV.—Euclid I. 33.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves also equal and parallel.

Let A B, C D (Fig. 28) be equal and parallel straight lines joined towards the same parts by the straight lines A C, B D. A C, B D are also equal and parallel.

Join B C; and because A B and C D are parallel the alternate angles A B C, B C D are equal; and because A B is equal to C D, and B C common to the two triangles A B C, B C D, and the angle A B C is equal to the angle B C D; therefore the base A C is equal to the base B D, and the triangle A B C to the triangle B C D, and the angle A C B opposite A B to the angle C B D opposite C D. And because the straight line D C meets the two A C, B D, making the alternate angles A C B, C B D equal to one another, A C is parallel to B D.

XXXVI.—Euclid I. 34.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, or divides them into two equal parts.

Let A B C D (Fig. 28) be a parallelogram of which B C is a diameter; the opposite sides and angles of the figure are equal to each other, and the diameter B C bisects it.

Because A B is parallel to C D, and B C meets them, the alternate angles A B C, B C D are equal; and because A C is parallel to B D, and B C meets them, the alternate angles A C D, C B D are also equal; wherefore the two triangles A C B, C B D have two angles A B C, B C A in one, equal to two angles B C D, C B D in the other, each to each; and one side B C common to the two triangles, which is adjacent to their equal angles. There-

fore the triangle B A C is equal to the triangle B D C, and the sides A B, A C to the sides C D, D B respectively, and the angle C A B to the angle B D C. Also, because the angle A B C is equal to B C D, and C B D to A C B, the whole angle A B D is equal to the whole angle A C D. Thus, all the opposite sides and angles are equal respectively, and it was shewn that the triangle A B C is equal to the triangle B C D; therefore, the diameter B C divides the parallelogram into two equal parts.

XXXVII.

Parallelograms upon the same base and between the same parallels are equal to one another.

Let the parallelograms A B C D, E B C F (Fig. 29) be upon the same base B C, and between the same parallels A F, B C; they shall be equal to each other.

Because A B C D is a parallelogram, A B is parallel and equal to D C; and because E B C F is a parallelogram, E B is equal and parallel to C F; therefore the two, A B, B E, are equal to the two D C, C F, respectively, and the angle A B E is equal to the angle D C F (Prop. 10), wherefore the triangle A B E is equal to the triangle D C F. Take the triangle A B E from the rectilineal figure A B C F, and from the same figure take the equal triangle D C F; therefore the remainders are

equal, that is the parallelogram A B C D is equal to the parallelogram E B C F.

XXXVIII.—Euclid I. 36.

Parallelograms upon equal bases and between the same parallels are equal to one another.

Let A B C D, E F G H (Fig. 30) be parallelograms upon equal bases, B C, F G, and between the same parallels, A H, B G. They are equal to each other.

Join B E, C H. And because B C is equal to F G and F G to E H, B C is equal to E H; but B C and E H are parallels joined towards the same parts by the straight lines B E, C H. Therefore (Prop. 35) B E, C H are themselves equal and parallel, and E B C H is a parallelogram, and it is equal to A B C D because it is on the same base B C, and between the same parallels. For a like reason it is equal to the parallelogram E H G F; wherefore E H G F is also equal to A B C D.

XXXIX.—Euclid I. 37.

Triangles upon the same base and between the same parallels are equal to one another.

Let the triangles A B C, D B C (Fig. 31) be upon the same base B C, and between the same parallels A D, B C; the triangle A B C is equal to D B C.

Produce A D both ways to E, F, and through B let B E be parallel to C A; and in like manner

let C F be parallel to C D. Therefore each of the figures E B C A, D B C F is a parallelogram; and E B C A is equal to D B C F, because they are upon the same base B C, and between the same parallels B C, E F; and the triangle A B C is half of the parallelogram E B C A, and the triangle D B C is half of the parallelogram D B C F (Prop. 37); wherefore the triangle A B C is equal to the triangle D B C.

XL.

Triangles upon equal bases and between the same parallels are equal to one another.

Let the triangles A B C, D E F (Fig. 32) be upon equal bases B C, E F, and between the same parallels B F, A D. The triangle A B C is equal to D E F.

Produce A D both ways to G and H, and let B G be parallel to A C, F H parallel to E D: then each of the figures G B C A, D E F H is a parallelogram, and they are equal to each other because they are upon equal bases B C, E F, and between the same parallels B F, G H; and the parallelogram G A B C is double of the triangle A B C, and D E F H is double of D E F; wherefore the triangle A B C is equal to D E F.

XLI.

If two triangles have one side, and the adjacent

angle of the one equal to one side and the adjacent angle of the other, and if the triangles are equal the other sides are also equal.

Let A B C, D E F (Fig. 32) be two triangles, having the angle B A C equal to the angle E D F, and the adjacent side A B equal to D E, and let the triangle A B C be equal to the triangle E D F. Then the other sides B C, A C are respectively equal to E F, D F.

Let the triangle A B C be super-imposed on D E F so that A B shall coincide with D E. Then, because the angle B A C is equal to E D F, the line A C will lie upon E F, and let the point C coincide with the point G in E F (produced if necessary). Join E G. Then the triangle D E G will be equal to the sum or difference of the triangles D E F, E F G, according as G falls without or within the base D F; that is, according as D G or A C is greater or less than D F; and, therefore, if A C is greater than D F, the triangle D E G or A B C is greater than D E F, and if A C is less than D F, the triangle A B C is less than D E F. Hence, conversely, if triangle A B C is not greater than D E F, A C is not greater than D F; and if A B C is not less than D E F, A C is not less than D F; and, therefore, if the triangle A B C is neither greater nor less than D E F, A C is neither greater nor less than D F; that is to

say, if A B C be equal to D E F, A C is equal to D F. And because A C or D G is equal to D F the point G will coincide with F, and the line E G with E F; wherefore E G or B C is equal to E F.

XLII.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Let the equal triangles A B C, D B C (Fig. 33) be upon the same base B C, and upon the same side of it; they are between the same parallels.

Let A E be drawn through A, parallel to B C cutting B D (produced if necessary) in E, and join A D, E C.

Then, because A E is parallel to B C, the triangle B E C is equal to B A C or B D C. And the equal triangles B E C, B D C have the base B C and the adjacent angle C B D common, wherefore (Prop. 41) B D is equal to B E, or the points D and E coincide, and, consequently, also the lines A D and A E; therefore, A D is parallel to B C.

XLIII.

Equal triangles upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let A B C, D E F be equal triangles on the straight line B F, having base B C equal to base

E F. The line A D joining their vertices will be parallel to B F.*

Let A G be parallel to B F, cutting D E (produced if necessary) in G. Join F G. Because E F is equal to B C, and A G is parallel to B F, the triangle G E F is equal to B A C (Prop. 40), and therefore to D E F. And the triangles D E F, G E F have the side E F, and angle D E F, common. Therefore (Prop. 41) the side E D is equal to E G or the points D and G coincide, and, consequently, also the lines A D, A G. Therefore A D is parallel to B F.

XLIV.—Euclid I. 41.

A parallelogram is double of the triangle on the same base and between the same parallels.

Let the parallelogram A B C D and the triangle E B C (Fig. 34) be upon the same base B C, and between the same parallels B C, A E. The parallelogram A B C D is double of E B C.

Join A C. Then the triangle A B C is equal to E B C, because they are on the same base and between the same parallels. But the parallelogram A B C D is double of A B C, wherefore also A B C D is double of E B C.

* The figure has been accidentally omitted, but it differs only from that of the last proposition in the bases of the two triangles being distinct.

XLV.—Euclid I. 43.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Let A B C D (Fig. 35) be a parallelogram, of which the diameter is A C; and E H, F G, the parallelograms about A C, that is, through which A C passes, and B K, K D the other parallelograms which make up the whole figure A B C D, which are therefore called the complements. B K shall be equal to K D.

Because A B C D is a parallelogram and A C its diameter, the triangle A B C is equal to A D C. And for the like reason, the triangle A E K is equal to A H K, and K G C to K F C; therefore the triangles A E K, K G C together are equal to the triangles A H K, K F C together. And the whole triangle A B C is equal to the whole A D C, wherefore the remaining complement B K is equal to the remaining complement K D.

XLVI.—Euclid I. 47.

In any right angled triangle the square which is described on the hypothenuse, or side opposite the right angle, is equal to the squares described upon the sides which contain the right angle.

Let A B C (Fig. 36) be a right angled triangle having the right angle B A C; the square described on the side B C is equal to the squares described upon B A, A C.

Let B D E C be the square described on B C; G B, H C the squares on the sides B A, A C; the line A L parallel to B D or C E. Join F C, B K.

Then because each of the angles B A C, B A G is a right angle A G, A C, are parts of the same straight line (Prop. 6); for the same reason B A, A H are in the same straight line. And because the angle D B C is equal to F B A, each of them being a right angle, add to each the angle A B C and the whole angle D B A is equal to the whole F B C. And because the two sides F B, B C, are equal to the two A B, B D, each to each, and the included angle F B C to A B D, the triangle F B C is equal to the triangle A B D. Now the parallelogram B L is double of the triangle B A D because they are upon the same base B D and between the same parallels B D, A L; and the square G B is double of the triangle F B C, because they are upon the same base E F and between the same parallels F B, G C. Therefore the parallelogram B L is equal to the square G B. In the same manner it may be shewn that the parallelogram C L is equal to the square C H; therefore the whole square B E is equal to the sum of the squares B G, C H.

XLVII.—Euclid I. 48.

If the square described upon one of the sides of a triangle be equal to the sum of the squares

described on the two other sides, the angle contained by these two sides is a right angle.

Let A B C (Fig. 37) be a triangle such that the square described on B C is equal to the sum of the squares on AB, A C; the angle B A C shall be a right angle.

Let C A D be a right angle on the other side of C A from A B; A D equal to A B; C D a straight line joining C and D. Then, because D A is equal to A B, the square of D A is equal to the square of A B: to each of these add the square of A C; therefore the squares of A D, AC are together equal to the square of A B, A C. But the square of D C is equal to the squares of A D and A C because C A D is a right angle, and the square of B C by hypothesis is equal to the squares of A B, A C. Wherefore the square of B C is equal to the square of D C, and therefore, also, B C is equal to D C. Therefore in the triangles C A D, C A B the sides C A, A D, C D are respectively equal to the sides C A, A B, C B; wherefore the triangles are wholly equal, and the angle C A D is equal to the angle C A B; but C A D is a right angle, wherefore C A B is also a right angle.

XLVIII.—Euclid II. 1.

If there be two straight lines one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the

rectangles contained by the undivided line and the several parts of the divided line.

Let A and B C (Fig. 38) be two straight lines whereof B C is divided into parts by the points D, E, etc. The rectangle contained by B C and A shall be equal to the sum of the rectangles contained by A and B D, D E, E C.

Let B H be the rectangle contained by A and B C, of which the sides B G, C H are each equal to A; and through D and E let D K, E L be parallel to B G or C H. Then the rectangle B H is equal to the sum of the rectangles B K, D L, E H. But B K is the rectangle contained by B D and B G or A, and each of the lines D K, E L, etc., is equal to B G or A, and therefore D L is the rectangle contained by D E and A; E H the rectangle contained by E C and A, and so on.

Wherefore the rectangle contained by B C and A is equal to the sum of the rectangles contained by A and each of the parts B D, D E, E C.

Cor. 1.—By making A equal to the whole line considered as divided into two parts, it appears that if a straight line be divided into any two parts, the square of the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.

Cor. 2.—In like manner making A equal to one of the parts, it appears that if a straight line be divided into any two parts, the rectangle contained by the

whole and one of the parts is equal to the rectangle contained by the two parts together with the square of the first mentioned part.

XLIX.—Euclid II. 4.

If a straight line be divided into any two parts the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line A B (Fig. 39) be divided into any two parts in C. The square of A B is equal to the squares of A C, C B and twice the rectangle contained by A C, C B.

Let A E be the square on A B, B D its diameter, C G F parallel to A D or B E, cutting B D in G, and through G let H G K be drawn parallel to A B or D E. Then, because C F and A D are parallel and A D meets them, the angle B G C is equal to B D A; but B D A is equal to D B A, because A D is equal to A B, being sides of a square; wherefore B G C is equal to G B C, and the side C G is equal to C B. But C B is also equal to G K, and C G to B K, because the figure C K is a parallelogram. Therefore the quadrilateral figure C K is equilateral. And because C G is parallel to B K, and C B meets them, the angles G C B, K B C are equal to two right angles; but C B K is a right angle, therefore, also, G C B is a right angle, and the angles opposite to them in

the parallelogram C K; viz., the angles C G K, G K B are also right angles. Wherefore the quadrilateral figure C K is also equiangular, and is therefore a square, and it is described on B C. For the same reason the rectangle H F is also a square, and it is described on H G, which is equal to A C; therefore C K, H F are the squares of B C, C A. And because the complement A G is equal to the complement G E, and A G is the rectangle contained by A C, C B (for G C is equal to C B), therefore G E is also equal to the rectangle A C, C B, and A G and E G together are equal to twice the rectangle A C, C B. But the whole figure A E is made up of the squares C K, H F, and the rectangles A G, G E; wherefore the whole square A E is equal to the squares of A C and B C and twice the rectangle A C, B C.

Cor.—From the demonstration it is manifest that the rectangles about the diameter of a square are likewise squares.

L.—Euclid II. 5.

If a straight line be divided into two equal and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line A B (Fig. 40) be divided into two equal parts at the point C, and into two

unequal parts at the point D; the rectangle A D, D B, together with the square of C D, is equal to the square of C B.

Let C E F B be the square on C B, B E its diameter; D H G, parallel to C E or B F, cutting B E in H; K L M passing through H parallel to B A or E F; A K parallel to C L or B M. Then, because the complement C H is equal to H F, to each of these add D M; therefore, the whole C M is equal to the whole D F. But C M is equal to A L, because A C is equal to C B; wherefore D F is equal to A L. To each of these add C H, and the whole A H is equal to D F and C H. But A H is the rectangle contained by A D, D B, for D H is equal to D B (Prop. 49. Cor.). Therefore the rectangle A D, D B is equal to C H and D F. To each of these quantities add L G which is equal to the square of C D; therefore the rectangle A D, D B, together with the square of C D, is equal to C H and D F and L G, that is to the square C F or the square of C B.

Hence it is manifest that the difference of the squares of two unequal lines A C, C D is equal to the rectangle contained by their sum and difference.

LI.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced,

together with the square of half the line bisected, is equal to the square of the line made up of the half and the part produced.

Let A B (Fig. 41) be bisected in C, and produced to D. The rectangle A D, D B, together with the square of C B, is equal to the square of C D.

Because the line A D is divided into parts at the points C, B, the rectangle A D, B D is equal to the rectangles A C, B D; C B, B D, and B D, B D; that is, to twice the rectangle B C, B D and the square of B D. To each of these add the square of B C. Then the rectangle A D, B D, together with the square of B C, is equal to the squares of B C and B D together with twice the rectangle B C, B D; that is (Prop. 49.), to the square of C D.

LII.

If a straight line be divided into any two parts, the squares of the whole line and of one of the parts is equal to twice the rectangle contained by the whole and that part together with the square of the other part.

Let A B (Fig. 39) be divided into any two parts in the point C, the squares of A B, B C are equal to twice the rectangle A B, B C together with the square of A C.

Because A B is divided into two parts at the point C (Prop. 48, *Cor.* 2), the rectangle A B, B C is equal to rectangle A C, B C and square of B C. Therefore

twice the rectangle A B, B C is equal to twice the rectangle A C, B C and twice the square of B C. To each of these add the square of A C. Then twice the rectangle A B, B C, together with the square of A C, is equal to the squares of A C, twice the square of B C, and twice the rectangle A C, B C; that is, to the squares of A C and B C, twice the rectangle A C, B C, and the square of B C again, or to the squares of A B, B C.

Hence it appears that the square of the difference of two straight lines is less than the sum of their squares by twice the rectangle of the two lines.

LIII.

If a straight line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line and of the square of the line between the points of section.

Let the straight line A B (Fig. 40) be divided at point C into two equal, and at D into two unequal parts. The squares of A D, D B are together double of the squares of A C, C D.

Because A D is divided into two parts at C, the square of A D is equal to the squares of A C, C D and twice the rectangle A C, C D; that is, to the squares of A C, C D and twice the rectangle B C, C D. To each of these add the square of D B. Then the squares of A D, D B are equal to the squares

of A C, C D together with twice the rectangle B C, C D and the square of B D. But because the straight line B C is divided into two parts at the point D, twice the rectangle B C, C D together with the square of B D is (Prop. 52) equal to the square of B C (that is of A C) and square of C D. Therefore the square of A D, D B is equal to twice the squares of A C, C D. That is to say, the sum of the squares of two lines, one of which is the sum of two straight lines, and the other their difference, is equal to twice the squares of the same lines.

LIV.

If a straight line be bisected and produced to any point, the square of the whole line thus produced and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let A B (Fig 41) be bisected in C and produced to D; the squares of A D, D B are double of the squares of A C, C D.

Because A D is divided into two parts at C, the square of A D is equal to the squares of A C, C D together with twice the rectangle A C, C D. To each of these add the square of D B. Then the squares of A D, D B are equal to the squares of A C, C D together with twice the rectangle A C, C D, that is twice the rectangle B C, C D and the

square of B D. But because C D is divided into two parts at B (Prop. 52) twice the rectangle C B, C D together with the square of B D, is equal to the squares of B C or A C and C D. Therefore the squares of A D, D B are equal to twice the squares of A C, C D.

LV.—Euclid II. 12.

In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted between the perpendicular and the obtuse angle.

Let A B C (Fig. 42) be a triangle having the obtuse angle A C B, and from either of the acute angles let A D be perpendicular on the opposite side B C produced. The square of A B is greater than the squares of A C, B C by twice the rectangle B C, C D.

Because B D is divided into two parts at C, the square of B D is equal to the squares of B C, C D together with twice the rectangle B C, C D. To each of these add the square of D A. Then the squares of B D, A D are equal to the squares of B C, C D, D A and twice the rectangle B C, C D.

But the square of B A is equal to the squares of B D, DA, because A D B is a right angle; and the square of A C is equal to the squares of C D and A D. Therefore the square of B A is equal to the squares of B C, A C and twice the rectangle A C, B C, or is greater than the squares of those lines by twice the same rectangle.

LVI.—Euclid II. 13.

In every triangle, the square of the side subtending any of the acute angles is less than the squares of the sides containing that angle by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

Let A B C be any triangle (Figs. 43, 42) and the angle at B one of its acute angles; A D the perpendicular on B C (produced if necessary) from the opposite angle. The square of A C is less than the squares of C B, C A by twice the rectangle C B, B D.

First let A D fall within the triangle A B C (Fig. 43), and because C B is divided into two parts at D, the squares of C B, B D (Prop. 52) are equal to twice the rectangle C B, B D and the square of D C. To each of these add the square of A D; therefore the squares of C B, B D, A D, are equal to twice the rectangle C B, B D, and

the squares of A D, D C. But the square of A B is equal to the squares of A D, D B, and the square of A C is equal to the squares of A D, D C. Therefore the squares of C B, A B are equal to twice the rectangle C B, B D and the square of A C. That is to say, the square of A C is less than the squares of A B, B C by twice the rectangle B C, B D.

Secondly, let A D fall without the triangle A B C (Fig. 42). Then, because the angle A D B is a right angle, the angle A C B is greater than a right angle, and therefore the square of A B (Prop. 55), is equal to the squares of A C, B C and twice the rectangle B C, C D. To these equals add the square of B C, and the squares of A B, B C are equal to the square of A C and twice the square of B C, and twice the rectangle B C, C D. But because B D is divided into two parts at C, the rectangle D B, B C is equal to the rectangle B C, C D and the square of B C; and the doubles of these are equal: therefore the squares of A B, B C are equal to the square of A C and double the rectangle D B, B C, that is the square of A C alone is less than the squares of A B, B C by twice the same rectangle.

Lastly, let the side A C be perpendicular to B C (Fig. 44); then is B C the straight line between the perpendicular and the acute angle at B, and it is manifest that the squares of A B, B C

are equal to the square of A C and twice the square of B C.

LVII.—Euclid III. 14.

Equal straight lines in a circle are equally distant from the centre, and those which are equally from the centre are equal to one another.

Let the straight lines A B, C D in the circle A B C D (Fig. 45) be equal to one another, they are equally distant from the centre, that is, the the perpendiculars from the centre on each of the lines are equal.

Let E be the centre of the circle, E F, E G perpendiculars to A B, C D. The straight lines A B, C D (Prop. 28) are bisected in F and G respectively, and are therefore the doubles of A F and C G respectively. A F is therefore equal to C G. And because A E is equal to E C, the square of A E is equal to the square of E C; but the squares of E F, F A are equal to the squares of A E; and the squares of E G, G C are equal to the square of E C; therefore the squares of E F, F A are equal to the squares of C G, G E, whereof the square of A F is equal to the square of C G, because A F is equal to C G. Therefore the remaining square of F E is equal to the remaining square of E G, and F E is equal to E G or the lines A B, C D are equally distant from the centre.

Next, if the lines A B, C D be equally distant

from the centre, that is, if F E be equal to E G, A B is equal to C D. For the same construction being made it may be shewn as before that A B is double of A F and C D of C G, and that the squares of E F, F A are equal to the squares of E G, G C, of which the square of F E is equal to the square of E G, because F E is equal to E G; therefore the remaining square of A F is equal to the remaining square of C G and A F to C G. Therefore the doubles of these or A B and C D are equal.

LVIII.—Euclid III. 15.

The diameter is the greatest straight line in a circle, and of all others that which is nearer to the centre is always greater than one more remote, and the greater is nearer to the centre than the less.

Let A B C D (Fig 46) be a circle of which the diameter is A D and centre E, and let B C, F G be any straight lines in the circle of which B C is nearer the centre than F G. A D is greater than B C and B C than F G.

Let E H, E K be perpendiculars to B C, F G, and join E B, E C, E F. Then because E A is equal to E B and E D to E C, but E B, E C are greater than B C, therefore A D is greater than B C.

And because B C is nearer to the centre than F G, E H is less than E K; but B C is double of B H, and F G double of F K; and the squares of

E H, H B are equal to the squares of E K, K F of which the square of E K is less than the square of E H, because E H is less than E K; therefore the square of B H is greater than the square of F K and the straight line B H greater than F K, and B C than F G.

Next let B C be greater than F G; B C shall be nearer to the centre than F G, that is, the same construction being made, E H shall be less than E K. Because B C is greater than F G, B H is greater than F K; and the squares of E H, H B, are equal to the squares of F K, K E, of which the square of B H is greater than the square of F K, because B H is greater than F K, therefore the square of E H is less than the square of E K, and the line E H than E K.

LIX.—Euclid III. 20.

The angle at the centre of a circle is double of the angle at the circumference upon the same base; that is, upon the same part of the circumference.

Let A B C (Fig. 47) be a circle, B E C an angle at the centre, and B A C an angle at the circumference, having the same circumference B C for their base; B E C is double of B A C.

Join A E, and produce it to the opposite side of the circumference in F. The angle B E C will be equal to the sum or difference of the angles B E F,

F E C, according as the point F lies within or without the circumference B C; that is, according as the centre of the circle falls within or without the angle B A C. And because E A is equal to E C the angle E A C is equal to E C A, and the external angle F E C, which is equal to E A C, and E C A is equal to double E A C. For the same reason the angle B E F is double of the angle B A F. Therefore B E C, the sum or difference of the angles B E F, F E C (as the case may be) is double of B A C, the sum or difference of the angles B A F, F A C.

LX.—Euclid III. 21.

The angles in the same segment of a circle are equal to one another.

Let A B C D (Fig. 48) be a circle; B A D, B E D angles in the same segment B A E D. The angles B A D, B E D are equal.

Let F be the centre of the circle, and first let the segment B A E D be greater than a semi-circle. Join B F, F D. Then, by Prop. 59, each of the angles B A D, B E D are half of the same angle B F D, and are therefore equal to each other.

But if the segment B A E D (Fig. 49) be not greater than a semi-circle, join A F and complete the diameter A F C, and join E C. Therefore the segment B A D C is greater than a semicircle, and the angles in it B A C, B E C are equal by the

first case. For the same reason, because C B E D is greater than a semicircle, the angles C A D, C E D are equal; therefore, the whole angle B A D is equal to the whole B E D.

LXI.—Euclid III. 22.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let A B C D (Fig. 50) be a quadrilateral figure in the circle A B C D, any two of its opposite angles are together equal to two right angles.

Join A C, D B; and take the angles A D B, B D C, into which any of the angles AD C is divided by the line D B. Then the angle B D C is equal to the angle B A C, because they are in the same segment B A D C; and the angle A D B is equal to A C B, because they are in the same segment A D C B; therefore the whole A D C is equal to the two C A B, A C B. To each of these add the angle A B C; therefore, the two A D C, A B C are equal to the three angles A C B, C A B, C B A of the triangle A B C; that is, to two right angles.

LXII.

In equal circles equal angles stand upon equal circumferences, whether they be at the centres or at the circumferences.

Let A B C, D E F (Fig. 51) be equal circles,

B A C, E D F; and B G C, E H F angles at the circumference and at the centre respectively. Then if either the angles at the circumference or at the centre are equal to each other, the other pair of angles will also be equal, because the angles at the centre are double of the corresponding angles at the circumference.

Let either BAC and EDF or BGC and EHF, and, therefore, both those pairs of angles be equal to each other. The circumference B K C is equal to E L F.

Because the circle A B C is equal to D E F, let these be superimposed on each other, so that the line G B coincides with the line H E; and because the angle B G C is equal to angle E H F, the line G C will coincide with H F, and the circumference B K C with E L F. Therefore B K C is equal to E L F.

LXIII.

In equal circles the angles which stand upon equal circumferences are equal, whether they be at the centres or the circumferences.

Let A B C, D E F (Fig. 51) be equal circles, B A C, E D F; and B G C, E H F the angles at the circumference and at the centre respectively, standing on equal circumferences B C, E F. The angle B G C is equal to E H F, and B A C to E D F.

Because the circles are equal, let the circle

A B C be superimposed on circle D E F so that the point B shall coincide with E. The line B G will coincide with E H; and, because the circumference B C is equal to E F, the point C will also coincide with F, and the line G C with H F. The angle B G C will then coincide with E H F, wherefore B G C is equal to E H F, and consequently the corresponding angle at the circumference B A C to E D F.

LXIV.—Euclid III. 28.

In equal circles equal straight lines cut off equal circumferences—the greater equal to the greater, and the less equal to the less.

Let A B C, D E F (Fig. 52) be equal circles, B C, E F equal straight lines in them. The two parts of the circumference B A C and B G C divided by the line B C, are equal to the parts E D F, E H F divided by the line E F—the larger to the larger and the less to the less.

Let K, L be the centres of the circles, and join K B, K C, L E, L F. Then because the circle A B C is equal to D E F, B K, K C are equal to D E, D F, and the third side B C is by the hypothesis equal to E F, wherefore the triangle B K C is equal to E L F and the angle B K C to E L F. But equal angles in equal circles stand on equal circumferences, wherefore B G C is equal to E H F and the remainder B A C to E D F.

LXV.

In equal circles equal circumferences are subtended by equal straight lines.

Let A B C, D E F (Fig. 52) be equal circles in which the circumferences B G C, E H F are equal, the straight line B C is equal to E F.

Because the circle A B C and circumference B G C are equal to the circle D E F and circumference E H F, the circle A B C and circumference E G C may be made to coincide with the circle D E F and circumference E H F. Therefore the points B and C respectively will coincide with the pionts E and F and the straight line B C with E F. Wherefore B C is equal to E F.

LXVI.—Euclid III. 31.

In a circle the angle in a semicircle is a right angle; in a segment greater than a semicircle, less,; and in a segment less than a semicircle, greater than a right angle.

Let A B C D (Fig. 53) be a circle of which the diameter is B C and centre E. Take any line C A dividing the circle into two parts of which A B C is greater and A D C less than a semicircle. Let D be any point in A D C and join A B, A E, A D, D C. The angle B A C is a right angle; the angle A B C in the segment A B C greater than a semicircle is less than a right angle; and the angle A D C in the segment A D C less than a semicircle is greater than a right angle.

Because E B is equal to E A the angle E B A is equal to B A E, and for a like reason the angle E C A is equal to the angle E A C; therefore the whole angle B A C is equal to the two A B C, A C B, or is half the three angles B A C, A B C, A C B, that is, half of two right angles, or is itself a right angle.

Next, because B A C is a right angle each of the other angles of the triangle B A C and therefore the angle A B C is less than a right angle.

And because A B C D is a quadrilateral figure in a circle (Prop. 61), the opposite angles A B C, A D C are together equal to two right angles; but A B C is less than a right angle, wherefore A D C is greater than a right angle.

LXVII.—Euclid III. 32.

If a straight line touch a circle and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle, are equal to the angles in the alternate segments of the circle.

Let the straight line E F (Fig. 54) touch the circle A B C D in the point B and let B D be a line dividing the circle into two segments D A B and D C B. The angles D B F, D B E are equal to the angles in the alternate segments; that is to say, the angle D B F to the angle in the segment on the other side of B D from B F, and the angle E B F to the angle in the segment on the other side of D B from E F.

Let A B be perpendicular to E F at B, and therefore a diameter of the circle. Let D be on the same side of the diameter with B F; take C any point in the segment adjacent to B F, and join A D, D C, C B. The angle A D B in a semicircle is a right angle, and therefore the other two angles of the triangle B A D, A B D are together equal to a right angle. But A B F is a right angle, and is therefore equal to the two B A D, A B D. Take away the angle A B D which is also part of A B F, and the remaining angle D B F is equal to B A D.

And because A B C D is a quadrilateral figure in a circle, the opposite angles B A D, B C D are together equal to two right angles. Therefore the angles D B E, D B F being also equal to two right angles are equal to B A D, B C D. And D B F has been proved to be equal to B A D; therefore the remaining angle D B E is equal to the angle B C D in the alternate segment of the circle.

LXVIII.—Euclid III. 35.

If two straight lines within a circle cut one another the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two straight lines A C, B C within the circle A B C D cut one another in the point E, the rectangle contained by A E, E C is equal to the rectangle contained by B E, E D.

First, if A C, C D intersect in the centre of the circle, it is evident that A E, E C, E B, E D being all equal, the rectangle A E, E C is equal to the rectangle B E, E D.

But let one of them B D (Fig. 55) pass through the centre F and cut the other A C which does not pass through the centre at right angles at the point E. Therefore (Prop. 28) D B bisects A C in E or A E is equal to E C. Join A F. And because the straight line B D is cut into two equal parts at point F and into two unequal at point E, the rectangle B E, E D together with the square of E F is equal to the square of F B or F A; but the squares of A E, E F are equal to the square of F A. Therefore the rectangle B E, E D together with the square of E F is equal to the squares of E F and A E. Take away the common square of E F and the rectangle B E, E D is equal to the square of A E, that is, to the rectangle A E, E C.

Next, let B D (Fig. 56) which passes through the centre cut the other A C which does not pass through the centre at E, but not at right angles. Let F G be perpendicular to A C. Therefore A G is equal to G C and as before the rectangle A E, E C together with the square of G E is equal to the square of A G. To each of these add the square of G F; therefore the rectangle A E, E C together with the squares of G E, G F is equal to the squares of A G and G F; that is, the rectangle A E, E C

together with the square of E F is equal to the square of A F or D F. But the square of D F is equal to the rectangle D E, E B together with the square of E F. Therefore the rectangle A E, E C together with the square of E F is equal to the rectangle D E, E B together with the same square and the rectangle A E, E C is equal to D E, E B.

Lastly, let neither of the lines pass through the centre (Fig. 57) and let H F E G be the diameter passing through the point of intersection E. Then by the last case the rectangle H E, E G is equal to each of the rectangles B E, E D and A E, E C and therefore those rectangles are equal to each other.

LXIX.—Euclid III. 36.

If from a point without a circle two straight lines be drawn whereof one cuts the circle and the other touches it, the rectangle contained by the whole line which cuts the circle and the part of it without the circle is equal to the square of the line which touches it.

If D be any point without a circle A B C and D C A, D B be two straight lines of which D C A cuts the circle in C and A and D B touches it in B, the rectangle D C, D A is equal to the square of D B.

First, let D C A pass through E the centre of the circle (Fig. 58) and join E B. Therefore E B D is a right angle (Prop. 29). And because the straight

line A C is bisected in E and produced to D (Prop. 51) the rectangle A D, D C together with the square of E C is equal to the square of E D. But the square of E D is equal to the squares of E B (or E C) and B D. Therefore the rectangle A D, D C and square of E C is equal to the squares of B D and E C and the rectangle A D, D C alone is equal to the square of B D.

But if D C A do not pass through the centre (Fig. 59) let E be the centre, E F perpendicular to D A and join E D, E C, E B. Because E F is perpendicular to A C (Prop. 28) it also bisects it, and F A is equal to F C. And because C A is bisected in F and produced to D the rectangle A D, D C together with the square of F C is equal to the square of F D. To each of these equals add the square of F E. Therefore the rectangle A D, D E together with the squares of C F, F E, is equal to the squares of D F and F E. But the squares of C F and F E are equal to the square of C E or E B, and the squares of D F and F E are equal to the square of D E. Therefore the rectangle D A, D C and together with the square of E B is equal to the square of D E, that is, to the squares of D B and E B. Therefore the rectangle D A, D C alone is equal to the square of D B.

Cor.—If from any point without a circle two straight lines be drawn cutting the circle, the rectangles contained by each straight line and the

part of it without the circle are equal to each other.

LXX.

If from a point without a circle be drawn two straight lines whereof one cuts the circle and the other meets it, and the rectangle contained by the whole line which cuts the circle and the part without the circle be equal to the square of the other line, the latter line touches the circle.

Let D be any point without the circle B C E (Fig. 59) whose centre is E, D C A a straight line cutting the circle in C and A; D B a line meeting it in B. Then if the rectangle contained by D C, C A is equal to the square of D B, D B touches the circle.

Because the square of D B is equal to the rectangle D C, C A, the square of D B is equal to the square of the line from D touching the circle and D B is equal to the same line, and because there cannot be two equal lines on the same side of the line passing through the centre, D B also coincides with the line touching the circle, that is, it touches the circle at the point B.

THE END.

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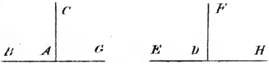

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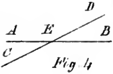
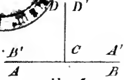


Fig 4



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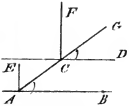

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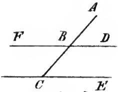


Fig. 8

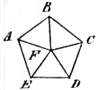
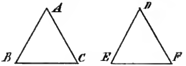


Fig 10



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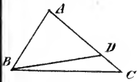

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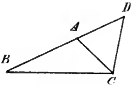

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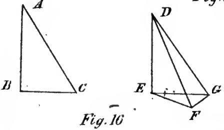

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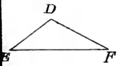
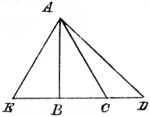


17



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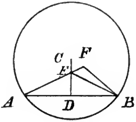

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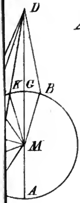

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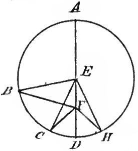

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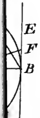

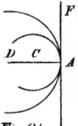

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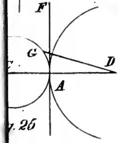

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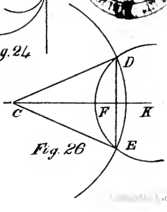

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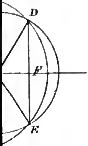

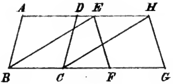

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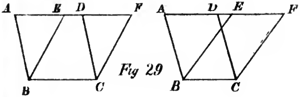

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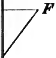

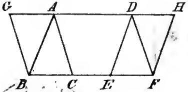

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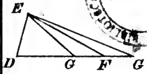

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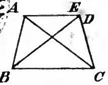

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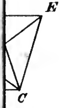

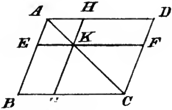

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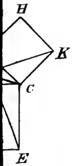

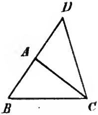

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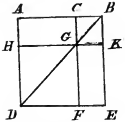

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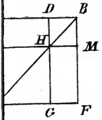

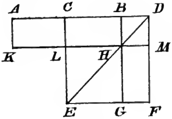

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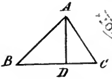

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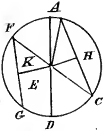

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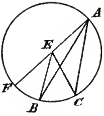

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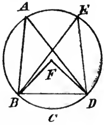

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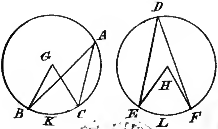

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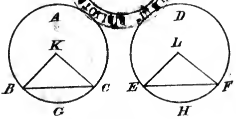

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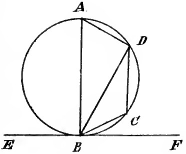

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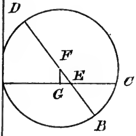

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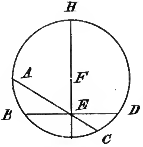

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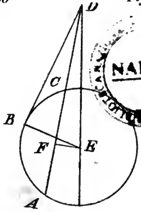

Fig. 59

34

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